IMAGE Problem 67-1

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Let n be a positive integer, and let J_n be the $n \times n$ matrix all of whose entries are equal to 1.

(a) Show that there exists a matrix $X \in M_n(\mathbb{Z})$ such that $X^2 + X = J_n$ if and only if $n = m^2 + m$ for some $m \in \mathbb{Z}$.

Solution. Suppose that $n = m^2 + m$ for some $m \in \mathbb{Z}$. Since $m^2 + m = (-m-1)^2 + (-m-1)$, we can assume, without loss of generality, that $m \geqslant 0$. Let X be the adjacency matrix of the Kautz graph on m symbols of dimension 2. Then $X^2 + X = J_n$, as required.

Conversely, suppose that $X^2 + X = J_n$ for some matrix $X \in M_n(\mathbb{Z})$. Let Y be the Jordan normal form of X. Then we can write $X = PYP^{-1}$ for some matrix P. It follows that $Y^2 + Y = P^{-1}J_nP$. Since the Jordan normal form is unique, and J_n is diagonalisable, we find that Y must be a diagonal matrix. Thus, X is diagonalisable. Now it is straightforward to show that the all-ones vector $\mathbf{1}$ is an eigenvector for X. Whence,

$$\begin{aligned} X^2\mathbf{1} + X\mathbf{1} &= J_n\mathbf{1} \\ m^2\mathbf{1} + m\mathbf{1} &= n\mathbf{1} \end{aligned}$$

for some integer m, as required.

(b) When n is of the form m^2+m, find the number of $n \times n$ zero-one matrices X which solve $X^2+X = J_n$.

Solution. First, observe that the diagonal of X must be 0. It suffices to show that the Kautz graphs of dimension 2 are (up to isomorphism) unique with respect to being graphs whose adjacency matrix X satisfies $X^2 + X = J_n$. This was shown by Gimbert in 1999 [1, Section 8]. Let $\Gamma = \text{Kautz}(m)$ with adjacency matrix X. Then the automorphism group $\text{Aut}(\Gamma)$ is $\text{Sym}(m + 1)$ (see [2]). Hence the number of $n \times n$ zero-one matrices X which solve $X^2 + X = J_n$ is $(m^2 + m)!/(m + 1)!$.

References

[1] J. Gimbert, *On digraphs with unique walks of closed lengths between vertices*, Australasian J. Combin. **20** (1999) pp. 77–90.

[2] L. Villar, *The underlying graph of a line digraph*, Discrete Applied Math. **37/38** (1992) 525–538.

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