Ratio bound (Lovász number) versus inertia bound

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Abstract

Matthew Kwan and Yuval Wigderson showed that for an infinite family of graphs, the Lovász number gives an upper bound of $O(n^{3/4})$ for the size of an independent set (where n is the number of vertices), while the weighted inertia bound cannot do better than $\Omega(n)$. Here we point out that there is an infinite family of graphs for which the Lovász number is $\Omega(n^{3/4})$, while the unweighted inertia bound is $O(n^{1/2})$.

1 An Example

We mostly use the notation of [2, §2] for association schemes. Let X be a finite set of size n. An *association scheme with d classes* is a pair $(X, \mathcal{R})$ such that

(i) $\mathcal{R} = \{R_0, R_1, \dots, R_d\}$ is a partition of $X \times X$,

(ii) $R_0 = \{(x, x) : x \in X\}$,

(iii) $R_i = R_i^T$, that is $(x, y) \in R_i$ implies $(y, x) \in R_i$,

(iv) there are numbers p_{ij}^k such that for any pair $(x, y) \in R_k$ the number of z with $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k.

Note that some authors call $(X, \mathcal{R})$ as defined above a *symmetric association scheme.* For relations R_i, the $\{0, 1\}$-adjacency matrices A_i are defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

As (i) holds, the matrices A_i are linearly independent, and as (iii) and (iv) hold, they generate a $(d+1)$-dimensional commutative algebra $\mathcal{A}$ of symmetric matrices, the *Bose-Mesner algebra.* Since the A_i commute, they can be diagonalized simultaneously and we find a decomposition of $\mathbb{C}^n$ into a direct sum of $d + 1$ eigenspaces of dimension f_j for $0 \leq j \leq d$. As the all-ones matrix J is in the span of A_i and has n as an eigenvalue of multiplicity 1, we may suppose that $f_0 = 1$. If $\{E_j : 0 \leq j \leq d\}$ is the basis of minimal idempotents of $\mathcal{A}$, then

$$f_j = \operatorname{rk} E_j = \operatorname{tr} E_j, \qquad \sum_{j=0}^{d} E_j = I, \qquad E_0 = n^{-1} J.$$

Define matrices P and Q by

$$A_j = \sum_{i=0}^{d} P_{ij} E_i, \qquad E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i.$$

Then $A_j E_i = P_{ij} E_i$ which shows that the P_{ij} are the eigenvalues of A_j. Also note that $Q_{0j} = f_j$ as $\mathrm{tr}(E_j) = f_j$.

For a subset Y of X with characteristic vector χ, define a vector $a = (a_i)$, the *inner distribution of* Y, by

$$a_i := \frac{1}{|Y|} \chi^T A_i \chi = \frac{1}{|Y|} |\{(x, y) \in Y \times Y \cap R_i\}|.$$

Delsarte's linear programming bound states that

$$(aQ)_j \geq 0$$

for all $0 \leq j \leq d$, see also Proposition 2.5.2 in [2].

We refer to [6] for details on the weighted ratio bound and the weighted inertia bound (also called Cvetković bound). It is well-known that the weighted ratio bound (also called Hoffman bound) is a special case of the Lovász number with equality in certain families of graphs. For graphs which correspond to a union of relations in an association scheme, Delsarte's linear programming bound for independent sets and the Lovász number are the same, see [7]. It is well-known that even the unweighted inertia bound sometimes gives a better bound on the independence number of a graph than the Lovász number. For instance, for the point graph of a generalized quadrangle of order (q, q^2), a graph with $(q^3 + 1)(q + 1)$ vertices, the unweighted inertia bound is $q^3 - q^2 + q$, while the Lovász number is $q^3 + 1$. Anurag Bishnoi asked if the inertia bound can also be asymptotically better than the Lovász number (as a parameter of the number of vertices n) [1]. The purpose of this note is to point out that there exists a graph on n vertices for which the Lovász number is $\Omega(n^{3/4})$, but the weighted inertia bound is $O(n^{1/2})$. In [3] Cameron and Seidel describe a 3-class association scheme which has the following P- and Q-matrices (follow the instructions in [4, page 2] together with [5] to obtain P and Q in a convenient manner):

$$P = \begin{pmatrix} 1 & 2^{2t}-1 & 2^{4t-2}+2^{3t-2}-2^{2t-1}-2^{t-1} & 2^{4t-2}-2^{3t-2}-2^{2t-1}+2^{t-1} \\ 1 & -1 & 2^{3t-2} - 2^{t-1} & -2^{3t-2} + 2^{t-1} \\ 1 & -1 & -2^{t-1} & 2^{t-1} \\ 1 & 2^{2t} - 1 & -2^{2t-1} - 2^{t-1} & -2^{2t-1} + 2^{t-1} \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & 2^{2t} - 1 & 2^{4t-1} - 3 \cdot 2^{2t-1} + 1 & 2^{2t-1} - 1 \\ 1 & -1 & -2^{2t-1} + 1 & 2^{2t-1} - 1 \\ 1 & 2^t - 1 & -2^t + 1 & -1 \\ 1 & -2^t - 1 & 2^t + 1 & -1 \end{pmatrix}$$

Hence, using $f_j = Q_{0j}$, the graph with adjacency matrix A_3 has eigenvalues

- $2^{4t-2} - 2^{3t-2} - 2^{2t-1} + 2^{t-1}$ with multiplicity 1,
- $-2^{3t-2} + 2^{t-1}$ with multiplicity $2^{2t} - 1$,

- 2^{t-1} with multiplicity $2^{4t-1} - 3 \cdot 2^{2t-1} + 1$,
- $-2^{2t-1} + 2^{t-1}$ with multiplicity $2^{2t-1} - 1$.

Hence, the unweighted inertia bound is

$$(2^{2t} - 1) + (2^{2t-1} - 1) = 3 \cdot 2^{2t-1} - 2.$$

The inner distribution a of an independent set Y of the graph has the form $a = (1, x, y, 0)$, where $|Y| = 1 + x + y$. Hence, the Lovász number is the solution to the linear program which maximizes $1 + x + y$ under the constraints that $(aQ)_j \geq 0$ for $0 \leq j \leq d$. As $(aQ)_3 \geq 0$, we find that

$$y \leq (x + 1) \cdot (2^{2t-1} - 1).$$

As $(aQ)_2 \geq 0$, we find that

$$(2^{2t-1} - 1)x + (2^t - 1)y \leq 2^{4t-1} - 3 \cdot 2^{2t-1} + 1.$$

Clearly, $x = 2^t - 1$ and $y = 2^t \cdot (2^{2t-1} - 1)$ maximizes $1 + x + y$. As $(aQ)_j \geq 0$ for all j for this solution, this is an optimal solution. Hence, the Lovász number of the graph is 2^{3t-1}. Together with [6], this shows the asymptotic incomparability of the Lovász number and the weighted inertia bound.

References

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