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On the Maximum Size of Erdős-Ko-Rado Sets in $H(2d+1,q^2)$

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Dedicated to Dieter Jungnickel on the occasion of his sixtieth birthday.

Abstract Erdős-Ko-Rado sets in finite classical polar spaces are sets of generators that intersect pairwise non-trivially. We improve the known upper bound for Erdős-Ko-Rado sets in $H(2d+1,q^2)$ for $d>2$ and d even from approximately q^{d^2+d} to q^{d^2+1}.

Keywords Erdős-Ko-Rado Theorem · Polar Space · Association Scheme · Linear Programming Bound

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1 Introduction

An *Erdős-Ko-Rado set* (EKR set) of generators in a finite classical polar space is a set of generators with pairwise non-empty intersection. For all finite classical polar spaces except $H(2d+1,q^2)$, with q a prime power, $d>2$ even,

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the largest EKR sets were classified in [5]. For $H(2d+1,q^2)$, $d>3$ even, the largest known EKR set consists of all generators on a point and has size $\prod_{i=0}^{d}(q^{2i+1}+1)\approx q^{d^2}$. On the other hand, the best known upper bound on EKR sets Y in $H(2d+1,q^2)$, $d\geq 3$ even, is approximately $|Y|\leq q^{d^2+d}$ (see e.g. [6]). The present paper improves this bound. Our main result is the following.

Theorem 1 *Let Y be an Erdős-Ko-Rado set of generators in $H(2d+1,q^2)$, $d\geq 2$ even. Then*

$$|Y|\leq \frac{nq^d-f_0(q^d-1)(1-c)}{q^{2d+1}+q^d+f_0(q^d-1)c}\approx q^{d^2+1}$$

where $n=\prod_{i=0}^{d}(q^{2i+1}+1)$, $f_0=q^2\begin{bmatrix}d+1\\1\end{bmatrix}\frac{q^{2d-1}+1}{q+1}$ *and* $c=\frac{q^2-q-1+q^{-2d+1}}{q^{2(d+1)}-1}$.

2 Association Scheme of $H(2d+1,q^2)$

We need some basic properties of the association scheme of $H(2d+1,q^2)$, with q a prime power, d even and $d>0$. A complete introduction to association schemes is e.g. given in [1, Ch. 2]. Note that contrary to [1, Ch. 2] indices in the present paper start with -1 and not with 0.

Definition 1 An *association scheme* with $d+1$ classes is a pair $(X,\mathcal{R})$, where X is a finite set and $\mathcal{R}=(R_{-1},R_0,\ldots,R_d)$ is a set of symmetric binary relations on X with the following properties:

1. $\{R_{-1},R_0,\ldots,R_d\}$ is a partition of $X\times X$.
2. R_{-1} is the identity relation.
3. There are integers p_{ij}^k such that for $x,y\in X$ with xR_ky there are exactly p_{ij}^k elements z with xR_iz and zR_jy.

The number $n_i:=p_{ii}^{-1}$ is called the *i-valency* of R_i. The total number of elements of X is

$$n:=|X|=\sum_{i=-1}^{d}n_i.$$

The relations R_i are described by their $\{0,1\}$-adjacency matrices $A_i\in\mathbb{C}^{n,n}$ defined by

$$(A_i)_{xy}=\begin{cases}1 & \text{if } xR_iy\\ 0 & \text{otherwise.}\end{cases}$$

There exist (see e.g. [1, p. 45]) positive semi-definite symmetric matrices $E_j \in \mathbb{C}^{n,n}$ with the properties

$$\sum_{j=-1}^{d} E_j = I, \tag{1}$$

$$E_{-1} = n^{-1}J, \tag{2}$$

$$A_j = \sum_{i=-1}^{d} P_{ij}E_i \tag{3}$$

and

$$E_j = \frac{1}{n}\sum_{i=-1}^{d} Q_{ij}A_i \tag{4}$$

where $P = (P_{ij}) \in \mathbb{C}^{d+2,d+2}$ and $Q = (Q_{ij}) \in \mathbb{C}^{d+2,d+2}$ are the *eigenmatrices* of the association scheme. Let f_j denote the rank of E_j. Let Δ_n denote the diagonal matrix $\operatorname{diag}(n_{-1},\dots,n_d)$ and let Δ_f denote the diagonal matrix $\operatorname{diag}(f_{-1},\dots,f_d)$. Then according to [1, p. 46, eq. 3]

$$Q = \Delta_n^{-1}P^t\Delta_f. \tag{5}$$

For integers $d, s \geq 0$ define

$$\begin{bmatrix} d \\ s \end{bmatrix} := \prod_{j=1}^{s} \frac{q^{2(d+1-j)}-1}{q^{2(s+1-j)}-1}.$$

Eisfeld calculated the matrix P in [2, Theorem 3.8] for classical polar spaces. For the association scheme of generators of $H(2d+1,q^2)$ his formula for all P_{ij} simplifies for $i = 0$ to

$$P_{0,s} := \begin{bmatrix} d \\ s+1 \end{bmatrix} q^{(s+1)^2} - \begin{bmatrix} d \\ s \end{bmatrix} q^{s^2}.$$

The multiplicities f_j are given in [1, Theorem 9.4.3]. In particular

$$f_0 = q^2 \begin{bmatrix} d+1 \\ 1 \end{bmatrix} \frac{1+q^{2d-2+1}}{1+q^{2d+1-1}} \cdot \frac{1+q^{2d+1-1}}{1+q} = q^2 \begin{bmatrix} d+1 \\ 1 \end{bmatrix} \frac{q^{2d-1}+1}{q+1}.$$

We are interested in the association scheme of generators of $H(2d+1,q^2)$. In this case X is the set of generators of $H(2d+1,q^2)$ with

$$n = \prod_{i=0}^{d}(q^{2i+1}+1)$$

and the definition of the relation R_i, $-1 \leq i \leq d$, is aR_ib if and only if $\dim(a \cap b) = d-i-1$ for all $a, b \in X$.

For this association scheme n_s is the number of generators of $H(2d+1, q^2)$ that meet a given generator in a $(d-s-1)$-dimensional subspace. A generator G of $H(2d+1, q^2)$ contains $\begin{bmatrix} d+1 \\ d-s \end{bmatrix}$ subspaces of dimension $d-s-1$. According to [3, Lemma 10] the number of generators in $H(2s+1, q^2)$ skew to a given generator is $q^{(s+1)^2}$. Hence we have $\begin{bmatrix} d+1 \\ d-s \end{bmatrix}$ possibilities to choose an $(d-s-1)$-dimensional subspace of G and $q^{(s+1)^2}$ generators that meet G exactly in this $(d-s-1)$-dimensional subspace. This yields

$$n_s = q^{(s+1)^2} \begin{bmatrix} d+1 \\ d-s \end{bmatrix} = q^{(s+1)^2} \begin{bmatrix} d+1 \\ s+1 \end{bmatrix}.$$

Together with (5) we obtain

$$Q_{s,0} = n_s^{-1} P_{0,s} f_0 \tag{6}$$

$$= \frac{f_0 \left(\begin{bmatrix} d \\ s+1 \end{bmatrix} q^{(s+1)^2} - \begin{bmatrix} d \\ s \end{bmatrix} q^{s^2} \right)}{q^{(s+1)^2} \begin{bmatrix} d+1 \\ s+1 \end{bmatrix}} \tag{7}$$

$$= f_0 \frac{q^{2(d-s)} - q - 1 + q^{-2s-1}}{q^{2(d+1)} - 1}. \tag{8}$$

In the following calculations we will make use of

$$Q_{-1,0} = f_0, \tag{9}$$

$$Q_{d-1,0} = f_0 \frac{q^2 - q - 1 + q^{-2d+1}}{q^{2(d+1)} - 1}, \tag{10}$$

and for $d-1 > s > -1$

$$(q^{2(d+1)} - 1) \frac{Q_{s,0} - Q_{d-1,0}}{f_0} = q^{2(d-s)} - q^{-2d+1} + q^{-2s-1} - q^2 \tag{11}$$

$$\geq q^4 - q^{-1} - q^2 > 0.$$

Let Y be a subset of X and χ the characteristic vector of Y. Put $y := |Y|$. The *inner distribution vector* $a = (a_{-1}, a_0, \ldots, a_d)$ of Y is defined by

$$a_i = \frac{1}{y} \chi^t A_i \chi.$$

Of course $a_{-1} = 1$ and $\sum_{i=-1}^{d} a_i = y$. Furthermore (4) implies that

$$(yaQ)_j = n \chi^t E_j \chi. \tag{12}$$

The matrices E_j are positive semidefinite, therefore

$$\chi^t E_j \chi \geq 0$$

for all $j \in \{-1, 0, \dots, d\}$. According to e.g. [5, p. 1295]

$$P_{i,d} = (-1)^{i+1} q^{2i^2+(1-2d)i+d^2}$$

and, if d is even, $P_{d,d}$ is the smallest value of all $P_{i,d}$. Together with the formulas (1), (2), and (3) these inequalities imply for $d \geq 2$ even

$$\begin{aligned}
0 &\leq n\chi^t \left(\sum_{i=1}^{d} (P_{i,d} - P_{d,d}) E_i \right) \chi \\
&= n\chi^t \left(A_d - P_{-1,d} E_{-1} - P_{0,d} E_0 \right) \chi - n P_{d,d} \chi^t \left(I - E_{-1} - E_0 \right) \chi \\
&= n\chi^t A_d \chi - n P_{d,d} \chi^t \chi - (P_{-1,d} - P_{d,d}) \chi^t J \chi - n (P_{0,d} - P_{d,d}) \chi^t E_0 \chi \\
&= n\chi^t A_d \chi - n P_{d,d} y - (P_{-1,d} - P_{d,d}) y^2 - y (P_{0,d} - P_{d,d}) (aQ)_0.
\end{aligned}$$

Let Y be an EKR set. By definition $\chi^t A_d \chi = 0$. Hence the previous inequality can be rewritten as

$$(P_{-1,d} - P_{d,d}) y + (P_{0,d} - P_{d,d}) (aQ)_0 \leq -n P_{d,d}.$$

As Y is an EKR-set, then $a_d = 0$ and hence

$$\begin{aligned}
a_{-1} &= 1, \text{ and} \\
a_{d-1} &= y - 1 - \sum_{i=0}^{d-2} a_i.
\end{aligned}$$

By (11) $Q_{i,0} - Q_{d-1,0} > 0$, hence

$$\begin{aligned}
(aQ)_0 &= \sum_{i=-1}^{d-1} Q_{i,0} a_i \\
&= Q_{-1,0} + Q_{d-1,0} a_{d-1} + \sum_{i=0}^{d-2} Q_{i,0} a_i \\
&= Q_{-1,0} + Q_{d-1,0} \left(y - 1 - \sum_{i=0}^{d-2} a_i \right) + \sum_{i=0}^{d-2} Q_{i,0} a_i \\
&= Q_{-1,0} - Q_{d-1,0} + Q_{d-1,0} y + \sum_{i=0}^{d-2} (Q_{i,0} - Q_{d-1,0}) a_i \\
&\geq Q_{-1,0} - Q_{d-1,0} + Q_{d-1,0} y.
\end{aligned}$$

Thus we obtain the inequality

$$(P_{-1,d} - P_{d,d}) y + (P_{0,d} - P_{d,d}) (Q_{d-1,0} y + (Q_{-1,0} - Q_{d-1,0})) \leq -n P_{d,d}.$$

This can be rewritten as

$$y \leq \frac{-nP_{d,d} - (P_{0,d} - P_{d,d})(Q_{-1,0} - Q_{d-1,0})}{P_{-1,d} - P_{d,d} + (P_{0,d} - P_{d,d})Q_{d-1,0}}$$
$$= \frac{nq^d - f_0(q^d - 1)\left(1 - \frac{q^2-q-1+q^{-2d+1}}{q^{2(d+1)}-1}\right)}{q^{2d+1} + q^d + f_0(q^d-1)\frac{q^2-q-1+q^{-2d+1}}{q^{2(d+1)}-1}}.$$

As $n \approx q^{(d+1)^2}$ and $f_0 \approx q^{4d}$, the largest term in the nominator is $nq^d \approx q^{d^2+3d+1}$, and the largest term in the denominator is q^{3d}. Hence the bound on y is $y \lessapprox q^{d^2+1}$, as stated in the theorem.

3 On the Limits of this Algebraic Method

The general idea is to take the inner distribution vector a and the $d+1$ linear inequalities

$$(aQ)_j \geq 0$$

and solve the corresponding linear optimization problem. For a given d we can apply a standard algorithm (e.g. some variant of the simplex algorithm, see [4]) and obtain the optimal solution. The matrix Q is given in [2], so it is easy to calculate an optimal solution for arbitrary d. For $d = 2$ this yields

$$y \leq q^5 + q^4 + q^3 + 1.$$

The present paper shows the same inequality. In [5] it is proven that the sharp upper bound is

$$y \leq q^5 + q^3 + q + 1.$$

For $d = 4$ linear optimization yields

$$y \leq q^{17} + 2q^{16} + 2q^{15} + q^{14} + q^{10} + 2q^9 + 3q^8 + 2q^7 + q + 1.$$

The present paper shows

$$y \leq q^{17} + 3q^{16} + 4q^{15} + 5q^{14} + 7q^{13} + 9q^{12} + 11q^{11} + 12q^{10} + 12q^9 + 9q^8 + 3q^7 - 7q^6 - 19q^5 - 35q^4 - 55q^3 - 77q^2 - 97q - 111.$$

The largest known examples for EKR sets of generators in $H(9, q^2)$ are the set of all generators through a point and the set of generators that meet a given generator in at least a plane, this selected generator included. Both examples have a size of approximately q^{16}.

Hence it seems reasonable to assume that this algebraic approach is not able to give the correct upper bound.

As a conclusion it does not seem worth to prove the best possible result with linear programming for general d for the following reasons.

1. The given proof is much easier than a proof of the linear programming bound.
2. The given bound is nearly as good as the expected bound through linear programming.
3. The best possible bound through linear programming is most likely wrong about the factor q for $d > 2$.

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