A New Upper Bound for Constant Distance Codes of Generators on Hermitian Polar Spaces of Type $H(2d-1, q^2)$

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In memory of Frédéric Vanhove.

Abstract. We provide new bounds for the maximum size of a set of generators of $H(2d-1, q^2)$ which pairwise intersect in codimension i by applying a multiplicity bound by C. D. Godsil. This implies a new bound on the maximum size of partial spreads of $H(2d-1, q^2)$, d even.

Mathematics Subject Classification (2010). 51A50; 05B25; 05C69; 51E23.

Keywords. Hermitian Polar Space; Association Scheme; Clique Number; Constant Distance Code; Partial Spread.

1. Introduction

This paper was motivated by the investigation of spreads and related structures in finite classical polar spaces. The first complete survey on spreads of polar spaces was done by J. A. Thas [19] in 1981. Later this problem was generalized to the study of partial spreads on polar spaces. A *partial spread* is a set of pairwise disjoint generators (maximal totally isotropic subspaces) of a polar space. From a graph theoretical point of view a partial spread is a clique of the disjointness graph of the generators of a polar space. The best result known to the author on the maximum size of partial spreads in $H(2d-1, q^2)$, d even, is due to J. De Beule, A. Klein, K. Metsch, and L. Storme [6].

The problem of the maximum size of partial spreads is a special case of the problem of the maximum size of *constant distance codes* of generators in $H(2d-1, q^2)$. Constant distance codes are of particular importance for random network coding as introduced in [14]. We refer to [17] for the general concept of constant distance codes of subspaces. For generators of Hermitian polar spaces constant distance codes are sets of subspaces which pairwise intersect in codimension i. Partial spreads are constant distance codes with

$i = d$. The only non-trivial upper bounds known to the author on these sets for general i were provided in the PhD thesis of F. Vanhove [21].

A Hermitian polar space $H(2d-1, q^2)$ is the geometry induced by a non-degenerate Hermitian form f of $V(2d, q^2)$, where $H(2d-1, q^2)$ consists of all totally isotropic subspaces with respect to the form f. We refer to [13, Ch. 23] for details. Our main concern are the combinatorial properties of $H(2d-1, q^2)$, so we will summarize these in the following. All maximal totally isotropic subspaces of $H(2d-1, q^2)$ have the same rank. These are called *generators*. The Hermitian polar space $H(2d-1, q^2)$ posses

$$\prod_{i=1}^{d} (q^{2i-1} + 1)$$

of them. Isotropic subspaces of rank 1 are called *points* and $H(2d-1, q^2)$ posses

$$\frac{(q^{2d-1} + 1)(q^{2d} - 1)}{q^2 - 1}$$

of them. The number of generators on a point of $H(2d-1, q^2)$ equals the number of generators of $H(2d-3, q^2)$. A generator of $H(2d-1, q^2)$ contains $(q^{2d} - 1)/(q^2 - 1)$ points. Let Y be a (partial) spread of $H(2d-1, q^2)$. Using the given combinatorial properties, double counting pairs (P, G) with $P \in G$ and $G \in Y$ yields

$$|Y| \leq q^{2d-1} + 1 \tag{1.1}$$

with equality if and only if Y is a spread. This bound is never reached for $d > 1$. In some sense this bound corresponds to the sphere packing bound for codes if we consider all generators on a point as a sphere.

In the following we list the previous results on (partial) spreads in $H(2d-1, q^2)$ known to the author that improve the bound of (1.1).

Theorem 1.2 (De Beule, Klein, Metsch, Storme [6]**).** *Let Y be a partial spread of $H(3, q^2)$. Then*

$$|Y| \leq \frac{1}{2}(q^3 + q + 2).$$

In particular, this bound is sharp for $q = 2, 3$.

Theorem 1.3 (De Beule, Klein, Metsch, Storme [6]**).** *Let Y be a partial spread of $H(2d-1, q^2)$, $d > 2$ even. Then*

$$|Y| \leq q^{2d-1} - q^{3d/2}(\sqrt{q} - 1).$$

The following theorem is stated for the more general concept of near polygons in [21]. The Hermitian polar space $H(2d-1, q^2)$ is a regular near $2d$-polygon of order (q^2, q), so we will only provide this theorem for this particular case.

Theorem 1.4 (Vanhove [21, Theorem 6.4.10]**).** *Let Y be a set of generators of $H(2d-1,q^2)$ such that all elements of Y pairwise intersect in codimension i odd. Then*

$$|Y| \leq 1 + q^i.$$

Our result is the following:

Theorem 1.5. *Let Y be a set of generators of $H(2d-1,q^2)$, $d > 1$, such that all elements of Y pairwise intersect in codimension i. Then*

$$|Y| \leq q^{2d-1} - q\frac{q^{2d-2}-1}{q+1}.$$

In particular, this is a bound on the maximum size of partial spreads in $H(2d-1,q^2)$.

First we will compare the case that Y is a partial spread (so $i = d$), d even, to the previous results in the following table.

d even	q	Best known bound	Theorem
2	2	6	1.2, 1.5
2	4	25	[4]
2	$\neq 4$	$\frac{1}{2}(q^3+q+2)$	1.2
4	$2, 3$	$q^{2d-1} - q\frac{q^{2d-2}-1}{q+1}$	1.5 (new)
4	> 3	$q^{2d-1} - q^{3d/2}(\sqrt{q}-1)$	1.3
> 4		$q^{2d-1} - q\frac{q^{2d-2}-1}{q+1}$	1.5 (new)

These bounds are sharp for $H(3,4)$ [9] and $H(3,9)$ [10]. They are not sharp for $H(3,16)$ [4].[1] For all other cases the sharpness of these bounds seems to be unknown.

For d odd a sharp upper bound of q^d+1 on the maximum size of partial spreads of $H(2d-1,q^2)$ was proven by F. Vanhove [20]. Examples reaching this bound were given by A. Aguglia, A. Cossidente, L. Ebert for $d = 3$ [1], and by D. Luyckx for $d > 3$ odd [15].

Now we will discuss the general case. If i is odd, then Theorem 1.4 gives a better bound for all i. In particular, for $i = 1$ it is well-known that the largest example is the set of all $q+1$ generators on a fixed subspace of rank $d-1$.[2] According to [12, Remark 4] there exists a constant-rank distance code with $i = 2$ and $q = 2$ in $H(2d-1,q^2)$ of size

$$\frac{q^{2d}-1}{q^2-1}.$$

[1] This result by Cimráková and Fack is due to an intelligent computer search. A purely combinatorial proof can be found in the PhD thesis of Linda Beukemann [2].

[2] Let Y be such an set of generators of maximum size. Theorem 1.4 yields $|Y| \leq q+1$ as an upper bound. Let $a, b \in Y$. Then $a \cap b$ has codimension 1. By the same argument as in [16, Lemma 5], all elements of Y contain $a \cap b$. This proves that Y is the set of all $q+1$ generators on a fixed subspace of codimension 1 respectively rank $d-1$.

For $q = 2$ this is one less than the bound of Theorem 1.5. Therefore, Theorem 1.5 is nearly sharp. In particular, in the case $q = 2$, $i = 2$ the bound is sharp for $d = 2$ by Dye [9] and $d = 3$ as shown in a yet unpublished result by Maarten De Boeck [7].

A similar application of Godsil's bound for near polygons known to the author is the upper bound $s^5 - s^3 + s - 1$ on the size of partial distance-2 ovoids in the generalized hexagon with parameter (s, s^3) by K. Coolsaet and H. Van Maldeghem [5]. As in [21, Theorem 6.4.10] this bound and the result of this paper could be stated in a unified way for near polygons with similar parameters. Unfortunately, according to [8, Theorem 3.4] there are no other near polygons with appropriate parameters.

2. Association Schemes

We need some basic properties of an association scheme of the so-called dual polar graph of rank d. A complete introduction to association schemes can be found in [3, Ch. 2].

Definition 2.1. Let X be a finite set. An *association scheme* with $d + 1$ classes is a pair $(X, \mathcal{R})$, where $\mathcal{R} = \{R_0, \ldots, R_d\}$ is a set of symmetric binary relations on X with the following properties:

1. $\mathcal{R}$ is a partition of $X \times X$.
2. R_0 is the identity relation.
3. There are numbers p_{ij}^k such that for $x, y \in X$ with xR_ky there are exactly p_{ij}^k elements z with xR_iz and zR_jy.

The number $n_i := p_{ii}^0$ is called the *i-valency* of R_i. The total number of elements of X is

$$n := |X| = \sum_{i=0}^{d} n_i.$$

The relations R_i are described by their *adjacency matrices* $A_i \in \mathbb{C}^{n,n}$ defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } xR_iy \\ 0 & \text{otherwise.} \end{cases}$$

Denote the all-one-matrix by J. There exist (see [3, p. 45]) idempotent Hermitian matrices $E_j \in \mathbb{C}^{n,n}$ (hence they are positive semidefinite) with the properties

$$\sum_{j=0}^{d} E_j = I, \qquad E_0 = n^{-1}J,$$

$$A_j = \sum_{i=0}^{d} P_{ij}E_i, \qquad E_j = \frac{1}{n}\sum_{i=0}^{d} Q_{ij}A_i,$$

where $P = (P_{ij}) \in \mathbb{C}^{d+1,d+1}$ and $Q = (Q_{ij}) \in \mathbb{C}^{d+1,d+1}$ are the so-called eigenmatrices of the association scheme. The P_{ij}'s are the eigenvalues of A_j. The multiplicity f_i of P_{ij} satisfies

$$f_i = \operatorname{rank}(E_i) = \operatorname{tr}(E_i) = Q_{0i}.$$

In this paper we consider the association scheme corresponding to the dual polar graph of $H(2d-1,q^2)$. Here X is the set of generators of the polar space and two generators a, b of $H(2d-1,q^2)$ are in relation R_i if and only if a and b intersect in codimension i.

In particular, for the dual polar graph $H(2d-1,q^2)$ [3, Theorem 9.4.3]

$$f_d = q^{2d}\frac{q^{1-2d}+1}{q+1} = q^{2d-1} - q\frac{q^{2d-2}-1}{q+1}.$$

The eigenvalues P_{ij} can be found in the literature (for example in [21, Theorem 4.3.6], or [18]). In particular,

$$\frac{P_{di}}{P_{0i}} = (-q)^{-i}.$$

Then by [3, Lemma 2.2.1],

$$Q_{id} = \frac{P_{di}}{P_{0i}} Q_{0d} = f_d(-q)^{-i}.$$

3. Proof of Theorem 1.5

We will use the following bound by C. D. Godsil [11, Theorem 3.5]. We refer to the original paper [11] for the definition of weight vectors.

Theorem 3.1 (Godsil [11, Theorem 3.5]**).** *Let X be a graph with $G = Aut(X)$ acting transitively on both its vertices and its edges. Let λ be an eigenvalue of the adjacency matrix of X such that all the weight vectors of X on λ are distinct. Then if X contains a clique on c vertices, the multiplicity m of λ is larger or equal to $c-1$ and if $m = c-1$, $\lambda = \frac{k}{1-c}$ (here k is the valency of X).*

In the next corollary we will restate this theorem for the special case of association schemes and the adjacency graph A_i. In this case Godsil's weight vectors correspond to the rows of the matrices Z_j defined by $Z_j Z_j^T = E_j$. All rows of Z_j are distinct if and only if $Q_{0j} \neq Q_{ij}$. The equation $\lambda = \frac{k}{1-c}$ corresponds to $P_{ji} = \frac{P_{0i}}{1-c}$. It can be easily checked using the identities given in [3, Lemma 2.2.1] that this equation holds if and only if $Q_{ij} = -1$. Hence, we can state the following corollary.

Corollary 3.2. *Let R_i be a relation of an association scheme $(X, \mathcal{R})$. Let Y be a subset of X of maximum size under the condition that all elements of Y are pairwise in relation R_0 or R_i. Let $j \in \{1, \dots, d\}$. Then*

$$|Y| \leq 1 + f_j,$$

if $Q_{0j} \neq Q_{ij}$*, and*

$$|Y| \leq f_j$$

if in addition $Q_{ij} \neq -1$*.*

Remark 3.3.

(a) Corollary 3.2 talks about association schemes with automorphism groups that do not necessarily act transitively on X. This is not problem, since the proof of Theorem 3.1 only uses, in Godsil's notation, the fact that the angels $\langle \underset{\sim}{w}_\lambda(i), \underset{\sim}{w}_\lambda(j)\rangle$ are all equal and that the weights of X on λ are all equal. These regularity properties are satisfied by all association schemes.

(b) It is easy to prove Corollary 3.2 directly by calculating the rank of the submatrix of E_j indexed by Y.

Proof of Theorem 1.5. Let Y be a set of generators of $H(2d-1, q^2)$ such that the generators of Y are pairwise in relation R_0 or R_i, $i > 0$. By Corollary 3.2,

$$|Y| \leq f_d = q^{2d}\frac{q^{1-2d}+1}{q+1} = q^{2d-1} - q\frac{q^{2d-2}-1}{q+1},$$

since

$$Q_{0d} = f_d \neq f_d(-q)^{-i} = Q_{id} \neq -1,$$

and

$$f_d = q^{2d-1} - q\frac{q^{2d-2}-1}{q+1}.$$

The assertion follows. □

Acknowledgements

The author would like to thank John Bamberg, Klaus Metsch, Frédéric Vanhove, and one anonymous referee for their helpful remarks. The author learned about the used bound during his visit at the University of Western Australia. Thus the author wants to express his special thanks to John Bamberg who made this research stay possible and did discuss several clique bounds with him.

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