Report on the GM-Switching Class of $Sp(6, 2)$

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Abstract

This is a report on the abundance of strongly regular graphs (SRGs) with parameters $(63, 30, 13, 15)$. One such graph is the collinearity graph of $Sp(6, 2)$. We computed all 13,505,292 graphs which can be obtained by applying Godsil-McKay (GM) switching with a bipartition with of type $(4, 59)$ at most 5 times to $Sp(6, 2)$. We provide data about automorphism group sizes, clique sizes, and coclique sizes for these graphs. Note that there seems to be billions of graphs which can be obtained from $Sp(6, 2)$ with GM switching, so this collection of data is very incomplete.

1 Introduction

> Strongly regular graphs lie on the cusp between highly structured and unstructured. For example, there is a unique strongly regular graph with parameters (36, 10, 4, 2), but there are 32548 non-isomorphic graphs with parameters (36, 15, 6, 6).
> Peter Cameron, Random Strongly Regular Graphs?

This is a short report on computations which I did over the last months. If you are unfamiliar with strongly regular graphs and collinearity graphs of polar spaces, we refer to the vast existing literature on both topics, e.g. Andries Brouwer's website[1] or [4]. Let us give a short description of the collinearity graph of $Sp(2d, q)$: Vertices are 1-dimensional subspaces of $\mathbb{F}_q^{2d}$. Two 1-dimensional subspaces are adjacent if they are perpendicular with respect to the bilinear form $x_1y_2 - x_2y_1 + \ldots + x_{2d-1}y_{2d} - x_{2d}y_{2d-1}$. For $(d, q) = (3, 2)$, this graph has 63 vertices, is 30-regular, two adjacent vertices have 13 common neighbors, and two non-adjacent vertices have 15 common neighbors. See [1, 6] for a description of Godsil-McKay (GM) switching with a focus on $Sp(2d, q)$. This report is not intended (or suitable) for publication and I can[2] reproduce all the data presented here in a few days. Why did I summarize my computations? The collinearity graph of the polar space $Sp(6, 2)$ is in many ways the smallest really interesting representative of

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[1] `https://www.win.tue.nl/~aeb/graphs/srgintro.html`

[2] With my computational resources at the time of writing.

the family of collinearity graphs coming from finite classical polar spaces. Thus it is a good toy model to investigate general behavior.

Some basic facts about $Sp(6,2)$: It has an automorphism group of size 1,451,520, clique number 7 and coclique number 7. SRGs with the same parameters as $Sp(6,2)$ have spectrum $(30, 3^{35}, -5^{27})$, clique number at most 7 and coclique number at most 9.[3]

Let $\mathcal{F}$ denote the family of all SRGs with parameters $(63, 30, 13, 15)$. Let $\mathcal{F}_{2d,q}$ denote the family of all SRGs with the same parameters as the collinearity graph of $Sp(2d, q)$. In the following we provide some questions on $\mathcal{F}$ and $\mathcal{F}_{d,q}$ and, based on the data, what I consider their most likely answer.

In the conclusion of [2], Bishnoi, Pepe and I are implicitly asking if graphs such as the collinearity graph of $Sp(2d, q)$ can be modified such that they are clique-free.[4] A far more specific question is if there exists a graph in $\mathcal{F}_{6,q}$ which is K_4-free. If one can show this for infinitely many q, then this essentially determines the asymptotic behavior of the Ramsey number $r(4, n)$ [7].[5] This is still too general, so we are stuck with the following:

Question 1. *Does $\mathcal{F}$ contain K_4-free graphs?*

Probable answer: **no**. Even a targeted threshold based search could only find an K_6-free graph in $\mathcal{F}$. Notice that most SRGs in $\mathcal{F}$ seem to be I_8-free. As Anurag Bishnoi pointed out to me, we currently only know that the Ramsey number $r(4, 8)$ is at least 56 [8]. Therefore the following is variation of the question above, even so the answer is still probably no:

Question 2. *Does $\mathcal{F}$ contain a graph which is K_4-free and I_8-free? In other words, does a graph in $\mathcal{F}$ imply that $r(4, 8) \geq 64$?*

In [5] I was wondering[6] if $|\mathcal{F}_{2d,q}|$ growths very fast, let's say hyperexponentially, in q or d. Similarly, Bill Kantor[7] is wondering if for any group G you can find a d and a q such that there exists an SRG Γ with the same parameters as the collinearity graph of $Sp(2d, q)$ and $\mathrm{Aut}(\Gamma) = G$. The following is another question in the same vibe:

Question 3. *Do almost all graphs in $\mathcal{F}_{2d,q}$ have a trivial automorphism group?*

There is some flexibility in this question[8], but the answer is probably **yes** in all possible interpretations of the question. We used nauty-traces, cliquer, a tiny C program, and standard GNU tools for this small investigation.

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[3] This follows from Hoffman's ratio bound.

[4] Cf. `https://anuragbishnoi.wordpress.com/2020/01/11/bound-on-ramsey-numbers-from-finite-geometry/`

[5] Cf. `https://anuragbishnoi.wordpress.com/2019/09/30/ramsey-numbers-from-pseudorandom-graphs/`

[6] In slightly more general and too specific form at the same time ...

[7] See his talk at the conference "Finite Geometry and Extremal Combinatorics", Delaware, 2019.

[8] The word "almost" only makes sense if we consider asymptotic behavior. We have two parameters d and q, so it is not clear.

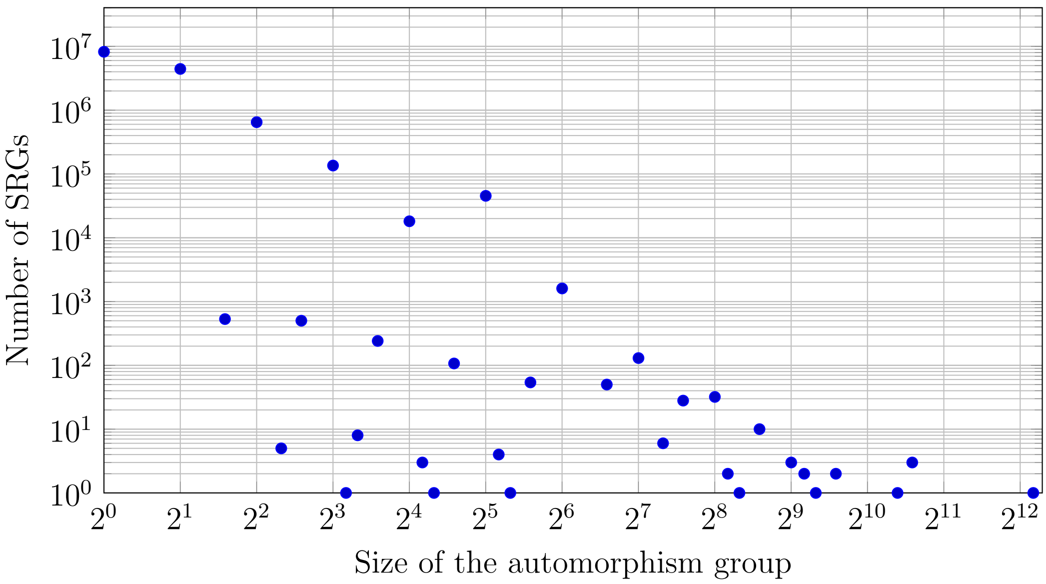
2 Tables

For the following data we used all graphs in $\mathcal{F}$ which can be obtained from the collinearity graph of $Sp(6,2)$ by applying a particular GM switching at most 5 times: we use a bipartition with one part of size 4 and one part of size 59. As mentioned in the abstract, this is only a tiny fraction of all graphs which can be obtained in this way. Indeed, here is a table of the number of graphs after applying the switching up to i times:

GM	0	1	2	3	4	5
New	1	2	52	3,275	254,097	13,247,865
Total	1	3	55	3,330	257,427	13,505,292

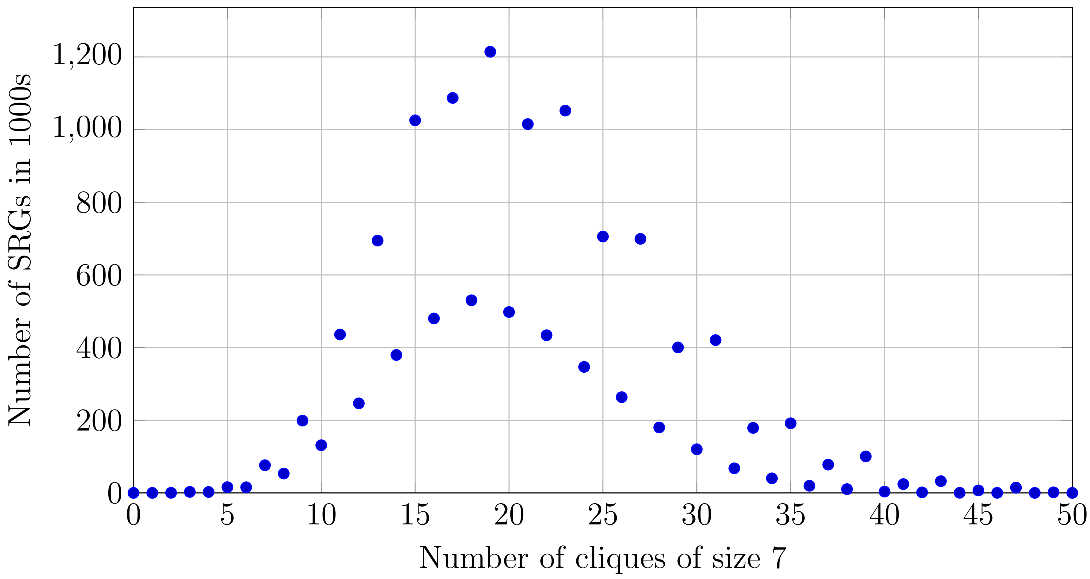
We start with a list of automorphism group size for the data set. Notice that powers of two occur far more often than other groups orders.

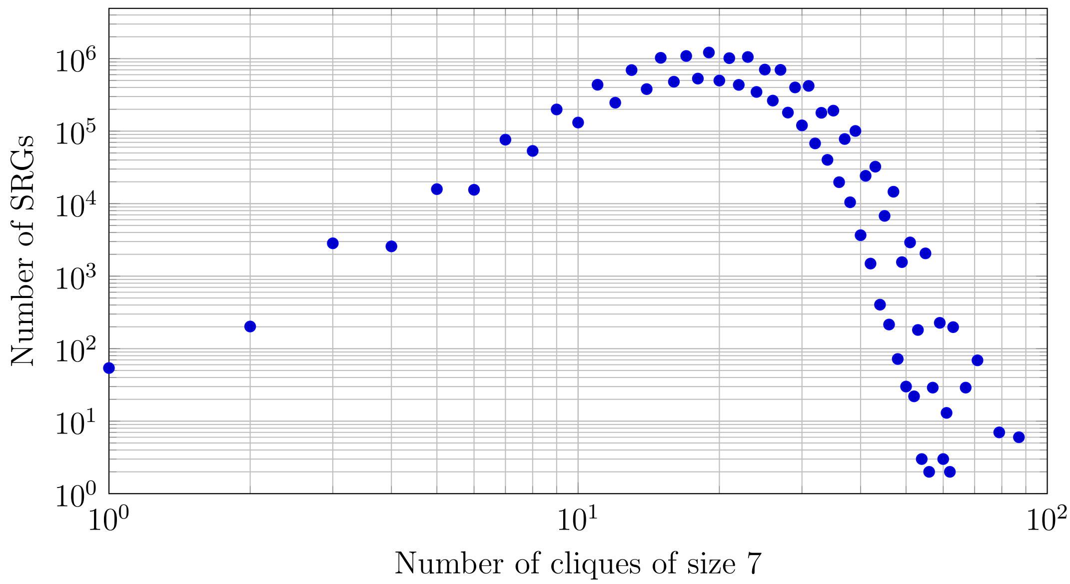
$\lvert G\rvert$	SRGs	$\lvert G\rvert$	SRGs	$\lvert G\rvert$	SRGs	$\lvert G\rvert$	SRGs
1	8,226,588	18	3	192	28	4,608	1
2	4,428,326	20	1	256	32	1,451,520	1
3	531	24	107	288	2		
4	648,049	32	45,390	320	1		
5	5	36	4	384	10		
6	501	40	1	512	3		
8	135,468	48	54	576	2		
9	1	64	1,605	640	1		
10	8	96	50	768	2		
12	241	128	130	1,344	1		
16	18,136	160	6	1,536	3		



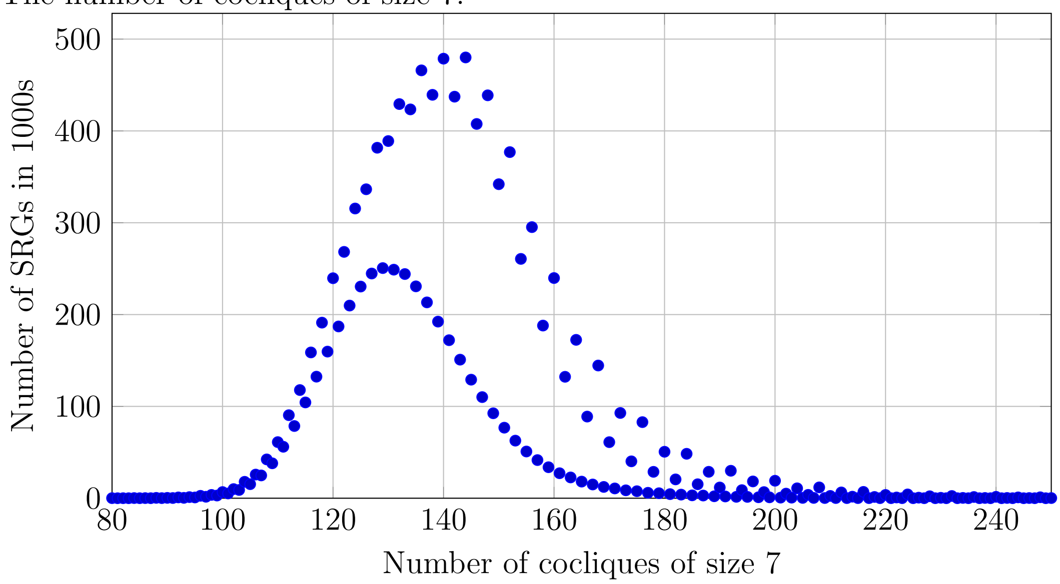
The number of cliques of size 7. We also found graphs in $\mathcal{F}$ with no cliques of size 6, but these examples are not reached in five steps. Note that there appear to be two or three curves which can be distinguished by parity conditions.

Cls	SRGs	Cls	SRGs	Cls	SRGs	Cls	SRGs
0	24	18	530,185	36	19,761	54	3
1	54	19	1,214,285	37	77,842	55	2,060
2	202	20	497,876	38	10,441	56	2
3	2,837	21	1,015,163	39	100,499	57	29
4	2,574	22	433,987	40	3,668	59	227
5	15,844	23	1,052,324	41	24,189	60	3
6	15,519	24	346,865	42	1,490	61	13
7	76,236	25	705,665	43	32,343	62	2
8	53,325	26	263,539	44	404	63	198
9	199,053	27	699,321	45	6,777	67	29
10	131,289	28	180,134	46	215	71	69
11	436,005	29	400,493	47	14,592	79	7
12	246,544	30	120,076	48	72	87	6
13	694,608	31	420,594	49	1,560	103	1
14	379,654	32	67,720	50	30	135	1
15	1,025,530	33	178,932	51	2,918		
16	480,205	34	40,152	52	22		
17	1,087,195	35	191,629	53	181		



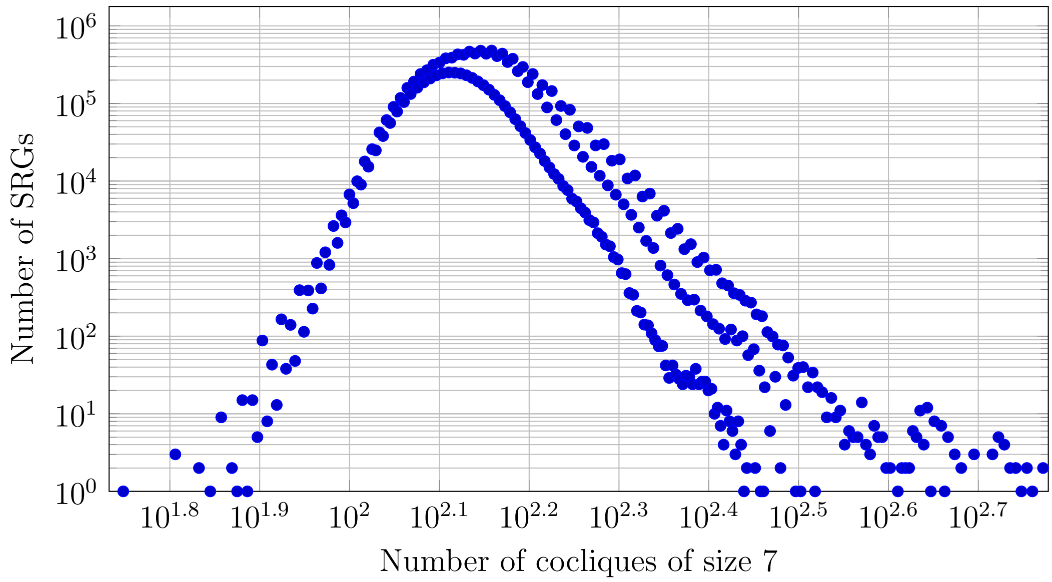


The number of cocliques of size 7:



As it is not obvious that most of the graphs with cocliques of size 7 do not have a colique of size 8, we also give table for cocliques of size 8. We also found graphs without cocliques of size 7, but these cannot be reached in five steps. Again, the plots suggests that there are three underlying curves distinguishable by parity conditions.

CCs	SRGs	CCs	SRGs	CCs	SRGs	CCs	SRGs	CCs	SRGs	CCs	SRGs
48	1	116	158,772	164	172,471	212	6,311	260	481	344	16
56	1	117	132,378	165	18,092	213	141	261	4	348	9
64	3	118	191,209	166	88,904	214	1,694	262	92	352	11
68	2	119	159,669	167	14,962	215	138	263	11	356	4
70	1	120	239,594	168	144,523	216	6,891	264	450	360	6
72	9	121	187,066	169	12,300	217	109	265	8	364	5
74	2	122	268,277	170	61,067	218	1,370	266	122	368	5
75	1	123	209,803	171	10,662	219	89	267	6	372	14
76	15	124	315,508	172	92,843	220	3,578	268	358	376	4
77	1	125	230,510	173	8,613	221	74	269	3	380	3
78	15	126	336,468	174	40,171	222	813	270	88	384	7
79	5	127	244,714	175	7,636	223	75	271	8	388	5
80	88	128	381,680	176	82,869	224	4,140	272	340	392	5
81	8	129	250,621	177	5,919	225	42	273	4	396	2
82	43	130	388,972	178	28,735	226	612	274	100	400	2
83	13	131	248,895	179	5,448	227	29	275	1	408	1
84	165	132	429,244	180	50,594	228	2,138	276	287	412	2
85	38	133	244,137	181	4,462	229	42	277	2	416	2
86	140	134	423,448	182	20,514	230	464	278	57	420	2
87	48	135	230,743	183	3,928	231	32	280	271	424	6
88	392	136	465,961	184	48,388	232	2,424	282	68	428	5
89	114	137	213,279	185	3,139	233	28	283	2	432	11
90	389	138	439,324	186	15,224	234	351	284	191	436	4
91	227	139	192,329	187	2,930	235	24	286	36	440	12
92	879	140	478,685	188	28,773	236	1,324	287	1	444	1
93	413	141	172,100	189	2,127	237	31	288	181	448	8
94	1,206	142	437,290	190	11,697	238	291	289	1	456	7
95	831	143	150,905	191	1,910	239	30	290	22	460	1
96	2,633	144	480,019	192	29,908	240	1,536	292	113	464	5
97	1,595	145	129,102	193	1,517	241	24	294	6	472	3
98	3,611	146	407,588	194	8,795	242	296	296	99	480	2
99	2,928	147	110,119	195	1,442	243	38	298	30	496	3
100	6,714	148	438,778	196	18,289	244	903	300	78	520	3
101	5,183	149	92,512	197	1,047	245	24	302	2	528	5
102	9,939	150	341,976	198	6,667	246	214	304	76	536	4
103	8,991	151	76,816	199	975	247	26	306	13	544	2
104	17,879	152	376,930	200	19,034	248	1,031	308	53	552	2
105	15,308	153	62,711	201	650	249	26	312	31	560	1
106	25,745	154	260,643	202	5,011	250	180	314	1	568	2
107	24,819	155	50,883	203	631	251	20	316	39	576	1
108	42,274	156	295,234	204	10,796	252	707	318	1	592	2
109	38,032	157	41,528	205	360	253	21	320	40	624	1
110	61,102	158	188,031	206	3,668	254	143	324	22	752	1
111	56,030	159	33,762	207	342	255	10	328	34		
112	90,436	160	239,785	208	11,806	256	718	330	1		
113	78,623	161	27,227	209	212	257	12	332	22		
114	117,801	162	132,234	210	2,511	258	125	336	19		
115	104,317	163	22,632	211	202	259	7	340	9		



Here the promised table with the number of cocliques of size 8.

CCs	SRGs	CCs	SRGs	CCs	SRGs	CCs	SRGs	CCs	SRGs
0	13,380,839	6	2,043	12	493	32	5	88	5
1	33,658	7	11	14	16	36	87	104	1
2	29,726	8	11,307	16	327	40	65		
3	3,356	9	8	18	5	44	2		
4	43,035	10	69	20	61	72	72		
5	55	11	16	24	28	76	2		

The 13,505,292 graphs can be found on the homepage of the author in Nauty's graph6 format: `http://math.ihringer.org/data.php`

References

[1] A. Abiad and W. H. Haemers. Switched symplectic graphs and their 2-ranks. *Des. Codes Cryptogr.*, 81(1):35–41, 2016.

[2] A. Bishnoi, F. Ihringer, and V. Pepe. A construction for clique-free pseudorandom graphs. *Combinatorica*, online first, 2020.

[3] Peter J. Cameron. Random strongly regular graphs? volume 273, pages 103–114. 2003. EuroComb'01 (Barcelona).

[4] X. L. Hubaut. Strongly regular graphs. *Discrete Math.*, 13(4):357–381, 1975.

[5] F. Ihringer. A switching for all strongly regular collinearity graphs from polar spaces. *J. Algebraic Combin.*, 46(2):263–274, 2017.

[6] F. Ihringer and A. Munemasa. New Strongly Regular Graphs from Finite Geometries via Switching. *Linear Algebra Appl.*, 580:464–474, 2019.

[7] D. Mubayi and J. Verstraete. A note on pseudorandom Ramsey graphs. *arXiv:1909.01461 [math.CO]*, 2019.

[8] S. P. Radziszowski. Small Ramsey numbers. *Electron. J. Combin.*, 1:Dynamic Survey 1, 30, 1994.