FINITE GEOMETRY INTERSECTING ALGEBRAIC COMBINATORICS

An Investigation of Intersection Problems Related to Erdős-Ko-Rado Theorems on Galois Geometries with Help from Algebraic Combinatorics

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PREFACE

I started my work on this thesis by reading the classification of nearly all Erdős-Ko-Rado sets (EKR sets) of generators in finite classical polar spaces by Valentina Pepe, Leo Storme, and Frédéric Vanhove. Their publication combines techniques from algebraic combinatorics with geometrical arguments and this heavily influenced my work. Methods from algebraic combinatorics are easy to apply, very general, and involve much theory. Contrary to this, finding geometrical counting arguments often involves much toying around with the problem until one finds a nice construction, which actually solves the problem, but might only work in a very specific case. In some cases, methods from algebraic combinatorics work better, in other cases, geometrical arguments seem to be preferable. Finally, in some cases combining both worlds is a powerful tool. As the reader will find out, this thesis contains examples for all of the above.

I am usually interested in families of objects which satisfy some nice condition. Consider the following.

- A family of 3-element subsets of $\{1,2,3,4,5,6\}$ which pairwise intersect in at least two elements. An example is

  $$\{1,2,3\},\ \{1,2,4\},\ \{1,2,5\},\ \{1,2,6\}.$$

- A family of 3-element subsets of $\{1,2,3,4,5,6\}$ which are pairwise disjoint. An example is

  $$\{1,2,3\},\ \{4,5,6\}.$$

- We put weights on $\{1,2,3,4,5,6\}$ such that the sum of all weights is zero. How many 3-element subsets of $\{1,2,3,4,5,6\}$ have non-negative weight? An example would be that we put the weight -5 on 1 and the weight 1 on the five other numbers. Then all 10 subsets with 3 elements, which contain the 1, have positive weight. The other 10 subsets have a negative weight.

There are many interesting question one can ask about these objects. This thesis is mostly concerned about proving upper and lower bounds as well as classifying the largest or smallest examples of these sets. This work investigates the vector space and polar space analogs of the generalizations of the problems given above.

This thesis is organized as follows. The necessary notation and some simple, often well-known results are introduced in Chapter 1. An overview over the state of the art of the considered Erdős-Ko-Rado problems is given in Chapter 3.

I started my work on my PhD with improving the best known upper bounds on EKR sets of generators in some Hermitian polar spaces together with Klaus Metsch. This resulted in Chapter 4, where we apply a technique from algebraic combinatorics, the linear programming bound, to the problem.

On our way back from the conference Combinatorics 2012 in Perugia on the bus to the airport, Klaus Metsch and I discussed the largest (d, t)-EKR sets on polar spaces. While we first thought that a similar approach as in Chapter 4 might be helpful, it turned out that a mixture of geometrical and algebraic techniques was a better way to classify these sets. While the basic ideas are simple, working out the complete arguments in detail was a very tiresome process during the last two years. This result is presented in Chapter 5.

In our attempts to prove an EKR theorem for $\mathsf{H}(9, q^2)$, Klaus Metsch also asked me about the maximum size of so-called cross-intersecting EKR sets. Based on eigenvalue techniques from algebraic combinatorics, this turned out to be a nice problem. Particularly, I was very happy that I found a problem, where I could bound the size of an object with Hoffman's bound and then classify all examples reaching this bound by eigenspace arguments and some geometry. Basically, I imitated the results by Valentina Pepe, Leo Storme, and Frédéric Vanhove on normal EKR sets. This homage to their work is written down in Chapter 6.

On the initiative of Klaus Metsch, I visited John Bamberg and the University of Western Australia from February to April 2013. There I collaborated with John Bamberg and Jan De Beule on weighted intriguing sets and the non-existence of ovoids in finite classical polar spaces. While our results are submitted and I did spend much time on them, they do not fit into the theme of this thesis. But I did discuss many techniques from algebraic combinatorics with John Bamberg, which resulted in Chapter 7; among other things a new bound on spreads in Hermitian polar spaces $H(2d-1, q^2)$, d even.

The last chapter of my thesis is about the Manickam-Miklós-Singhi conjecture, a problem closely related to EKR theorems. Simeon Ball mentioned this problem to me on a short car drive March 2014 in Ghent. I found the problem interesting, worked on it for three weeks (I did barely do anything else during that time), and got Chapter 8 as a reward, which solves the problem for most finite vector spaces.

Then there is also Chapter 2. There I collected all the inequalities I calculated during my PhD to approximate numbers in finite geometries. At least one inequality there is noteworthy, because it is a very decent approximation of the number of generators in a finite classical polar space.

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Finally, I would like to honor two great persons who deserve my thanks, but are no longer here to receive it.

> Frédéric Vanhove (1984–2013), an excellent mathematician whose scientific work still inspires me.

> Konrad Macholdt (1987–2014), an old friend and brilliant political activist.

PUBLICATIONS AND JOINT WORK

The author did work on several publications during his time at the Mathematisches Institut der JLU Gießen [5, 53, 50, 48, 52, 49, 51]. Five of these are the basis of this thesis: [53] for Chapter 4, [52] for Chapter 5, [49] for Chapter 6, [50] for Chapter 7, [51] for Chapter 8.

Two of the publications relevant for this thesis [53, 52] are joint work with Klaus Metsch.

The publications [53, 48, 50] are accepted or published at the time of writing. The publications [49, 51] are submitted. The publications [5, 52] are still in preparation.

These references and remarks will not be repeated in the specific chapters.

NEW RESULTS

Here we present some of the new results of this thesis, which the author considers noteworthy. For the sake of simplicity, this list simplifies some of the results. We refer to the specific chapters for the used notation.

- Lemma 2.7. Let $\alpha > 1$ be a real number with $\alpha \log(1+q^{-e-1}) \leqslant \log(1+q^{-e})$. Let x be the number of generators of a polar space of rank d and type e. Then

$$x \cdot q^{-de-\binom{d}{2}} = \prod_{i=0}^{d-1}(1+q^{-e-i}) \leqslant \left(1+q^{-e}\right)^{\frac{\alpha}{\alpha-1}}.$$

- Theorem 4.1. An EKR set of generators on $\mathsf{H}(2d-1,q^2)$, d odd, has at most approximately $\approx q^{(d-1)^2+1}$ elements.

- Theorem 5.1. The largest (d,t)-EKR set of generators of a finite classical polar space is the following.
  (a) **Case** t **even.** A d-junta if one of the following conditions is satisfied:
    (i) $t \leqslant \sqrt{\frac{8d}{9}} - 2$, $q \geqslant 2$,
    (ii) $t \leqslant \sqrt{\frac{8d}{5}} - 2$, $q \geqslant 3$.
  (b) **Case** t **odd.** A $(d-1)$-junta if one of the following conditions is satisfied:
    (i) $t \leqslant \sqrt{\frac{8d}{9}} - 2$, $q \geqslant 2$,
    (ii) $t \leqslant \sqrt{\frac{8d}{5}} - 2$, $q \geqslant 3$.

- Corollary 5.13. The largest $(d,2)$-EKR set of generators of a finite classical polar space is either a d-junta or corresponds to a $(3,2)$-EKR set.

- Theorem 5.34. A (d, t)-EKR set of generators of a finite classical polar space of type e over a finite field of order $q > 2$ has
  - at most approximately $q^{t(d-t-1)+\binom{t}{2}+te}$ elements if t odd or $e \geqslant 1$.
  - at most approximately $q^{t(d-t-1)+\binom{t+1}{2}}$ elements if t even and $e \leqslant 1$.
- Corollary 6.2, Theorem 6.4, Theorem 6.14, Theorem 6.16, and Theorem 6.19. A classification of the largest cross-intersecting EKR sets (Y, Z) of generators of finite classical polar spaces with respect to $|Y| \cdot |Z|$ with the exception of $H(2d-1, q^2)$, $d > 2$.
- Theorem 6.17. A cross-intersecting EKR set (Y, Z) of generators of on $H(2d-1, q^2)$, $d > 3$, satisfies approximately $\sqrt{|Y| \cdot |Z|} \leqslant q^{d^2-2d+2}$.
- Theorem 7.4. An $\{i\}$-clique of generators on $H(2d-1, q^2)$ has at most size
$$q^{2d-1} - q\frac{q^{2d-2}-1}{q+1}.$$
- Theorem 8.1. Let V be an n-dimensional vector space. Let f be a weighting of the points of V such that the sum of all weights is zero. Then for sufficiently large q and $n \geqslant 2k$, the number of nonnegative k-dimensional subspaces of V is at least $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. If this bound is tight, then the set of nonnegative k-dimensional subspaces is the set of k-dimensional subspaces containing a fixed point.
- Lemma 8.14. There is no need for an analog of Theorem 8.1 for $k \leqslant n < 2k$ as in this case the problem can be reduced to the case $2k \leqslant n$.

CONTENTS

1 PRELIMINARIES

The aim of this chapter is to fix some notation from linear algebra, (algebraic) graph theory, association schemes, and finite geometry. The author assumes that the reader is familiar with these topics.

1.1 LINEAR ALGEBRA

Let A be an $(n \times m)$-matrix over some field K. We say that A is a *quadratic matrix* if $n = m$. We denote the transpose of A by A^T. If K is the field $\mathbb{C}$ of the complex numbers, then we denote the complex conjugate of A by $\overline{A}$. We say that A is a *Hermitian matrix* if $\overline{A} = A^T$ and $K = \mathbb{C}$. We say that A is a *symmetric matrix* if $A = A^T$.

There are some special matrices and vectors for which we shall fix our notation in the following. Notice that the size of the following objects will be always clear from the context.

(a) We denote the *identity matrix* by $I \in K^{n \times n}$, i.e. a quadratic matrix where all entries on the diagonal are 1, all other entries are 0.

(b) We denote the *all-ones matrix* by $J \in K^{n \times n}$, i.e. a quadratic matrix where all entries equal 1.

(c) We denote the *all-ones vector* by $\mathbf{j} \in K^n$, i.e. the vector where all entries are 1.

(d) We denote the i-th vector of the canonical basis by $e_i \in K^n$, i.e. the i-th entry of e_i is 1, all the other entries are 0.

(e) We say that two vectors $\chi, \psi \in \mathbb{C}^n$ are *orthogonal* if $\overline{\chi}^T \psi = 0$.

(f) We define the *Euclidean norm* $|\chi|$ by $|\chi| = \sqrt{\chi^T \chi}$.

(g) The finite field with q elements is denoted by $\mathbb{F}_q$.

(h) The Kronecker delta δ_{ij} is 1 if $i = j$ and 0 otherwise.

Recall the following basic results from linear algebra on Hermitian matrices, in particular, on real symmetric matrices.

Lemma 1.1. *A Hermitian* $(n \times n)$*-matrix has (with multiplicities)* n *real eigenvalues.*

Lemma 1.2. *Let A be a Hermitian matrix. Then there exists an orthonormal basis of eigenvectors of A.*

Let $V_0, \dots, V_d$ be the eigenspaces of a Hermitian $(n \times n)$-matrix A. We denote the dimension of V_i by f_i. Let $W_i = \{w_1, \dots, w_{f_i}\}$ be an orthonormal basis of V_i. Let M_i be the matrix which has the elements of W_i as it columns. Define the $(n \times n)$-matrix E_i as $M_i M_i^T$.

The basis W_i is orthonormal, so we have $M_i^T M_i = I$. Hence,

$$E_i^2 = (M_i M_i^T)^2 = M_i M_i^T = E_i, \tag{1.1}$$

and E_i is idempotent. Let χ be a vector of $\mathbb{C}^n$. The mapping $\chi \mapsto E_i\chi$ can be seen as the orthogonal projection of χ onto the eigenspace V_i. A simple calculation shows

$$\sum_{i=0}^{d} E_i \chi = \chi. \tag{1.2}$$

This implies $\sum_{i=0}^{d} E_i = I$. If we denote the eigenvalue of the eigenspace V_i by λ_i, then we have the spectral decomposition theorem for Hermitian matrices.

Theorem 1.3. *Let A be a Hermitian matrix. Let $\lambda_0, \dots, \lambda_d$ be its distinct eigenvalues. Then there exist idempotent matrices $E_0, \dots, E_d$ such that*

$$A = \sum_{i=0}^{d} \lambda_i E_i.$$

1.2 GRAPH THEORY

The notation used in this thesis on graphs is mostly a mixture of [13] and [38]. A *graph* Γ is a pair $(X, \sim)$ where $\sim$ is a symmetric binary relation on the *vertex set* X. We say that $\sim$ is the *adjacency relation* of Γ. Sometimes we will view $\sim$ as a subset of $X \times X$. We say that x and y are *adjacent* or *neighbors* if $x \sim y$. We say that x and y are *non-adjacent* if not $x \sim y$. We say that the graph Γ is a *simple graph* if x is non-adjacent to itself for all vertices x. Two graphs $\Gamma_1 = (X, \sim_1)$ and $\Gamma_2 = (Y, \sim_2)$

are isomorphic if there exists a bijection f from X to Y that preserves adjacent, i.e. $x \sim_1 y$ if and only if $f(x) \sim_2 f(y)$.

A graph $\Gamma' = (Y, \sim_2)$ is called a *subgraph* of $\Gamma = (X, \sim)$ if Y is a subset of X and $\sim_2$ is a subset of $\sim$. We say that Γ' is an *induced subgraph* of Γ if $x \sim_2 y$ for all $x, y \in Y$ if and only if $x \sim y$. In particular, an induced subgraph is determined by its vertex set.

If all vertices Y of an induced subgraph Γ' of Γ are pairwise adjacent, then we call Y a *clique*. If there exists no clique Y' of Γ with $Y \subseteq Y'$ and $Y \neq Y'$, then we call Y a *maximal clique*. If there exists no clique Y' of Γ with $|Y| < |Y'|$, then we call Y a *maximum clique* and say that $|Y|$ is the *clique number* of Γ. Many authors denote the clique number of a graph Γ by $\omega(\Gamma)$.

If all vertices Y of an induced subgraph Γ' of Γ are pairwise non-adjacent, then we call Y an *independent set* or a *coclique*. If there exists no independent set Y' of Γ with $Y \subseteq Y'$ and $Y \neq Y'$, then we call Y a *maximal independent set*. If there exists no independent set Y' of Γ with $|Y| < |Y'|$, then we call Y a *maximum independent set* and say that $|Y|$ is the *independence number* of Γ. Many authors denote the independence number of a graph Γ by $\alpha(\Gamma)$.

One classical problem in combinatorial graph theory is to prove bounds on $\alpha(\Gamma)$ and $\omega(\Gamma)$. Finding good bounds on these parameters or finding large induced subgraphs with the desired property for particular graphs is, as we shall see in this thesis for a few particular examples, often a very challenging task.

Let a *path* of length d be a sequence $(x_0, \ldots, x_d)$ of $d+1$ vertices with $x_i \sim x_{i+1}$ for all i, $0 \leqslant i < d$. We define the *distance* $d(x, y)$ between two vertices x, y as the minimal length of a path $(x_0, \ldots, x_d)$ with $x_0 = x$ and $x_d = y$. If there exists no such sequence, then we set $d(x, y) = \infty$. If $d(x, y) < \infty$ for all vertices x, y of a graph Γ, then Γ is a *connected* graph. Notice that $d(x, x) = 0$ and $d(x, y) = 1$ if $x \sim y$. The *diameter* of a graph Γ is the maximum distance that occurs between two vertices of Γ. For given x we denote the set of vertices in Γ at distance i from x by $\Gamma_i(x)$.

The *degree* of a vertex is $|\Gamma_1(x)|$. A graph is called a *regular graph* or k-*regular graph* if it is simple and all vertices have degree k.

Let d be the diameter of a simple graph Γ. If there exist constants $c_1, \ldots, c_d$, $a_0, \ldots, a_d$, $b_0, \ldots, b_{d-1}$ such that for all vertices x, y of Γ

(a) $a_i = |\Gamma_i(x) \cap \Gamma_1(y)|$ where $i = d(x, y)$,

(b) $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$ where $i = d(x, y) < d$,

(c) $c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$ where $i = d(x, y) > 0$,

then Γ is a *distance-regular graph* with the *intersection array*

$$\{c_1, \ldots, c_d; a_0, \ldots, a_d; b_0, \ldots, b_{d-1}\}.$$

Obviously, all distance-regular graphs are regular graphs. We want to remark that the intersection arrays of nearly all known families of distance-regular graphs have so-called *classical parameters* and can be described by just four parameters [13, Ch. 6]. Also notice that a distance-regular graph is not uniquely determined by its intersection numbers, e.g. the so-called disjointness graph of lines of the symplectic polar space $W(3, 3)$ and the disjointness graph of lines of the parabolic polar space $Q(4, 3)$ have the same parameters, but are not isomorphic [13, p. 279].

If a distance-regular graph Γ has at most diameter 2, then Γ is called a *strongly regular graph*. For a strongly regular graph there exist numbers (n, k, λ, μ) such that Γ has n vertices, Γ is k-regular, two adjacent vertices have λ common neighbors, and two non-adjacent vertices have μ common neighbors, then (n, k, λ, μ) are the parameters of the strongly regular graph Γ.

A graph is called *distance-transitive graph* if the automorphism group of a graph acts transitively on pairs of vertices at the same distance. It is easy to see that all distance-transitive graphs are distance-regular graphs. As it is always the case with regularity properties and analog symmetry properties, distance-regularity is weaker than distance-transitivity. There exist plenty of distance-regular graphs that are not distance-transitive.[1]

1 There are plenty examples even for strongly-regular graphs, see [54] for a classification of some strongly regular graphs with parameters $(57, 24, 11, 9)$. The authors provide approximately 10^{10} non-isomorphic distance regular graphs with these parameters and nearly all of them have a trivial automorphism group.

An *adjacency matrix* A of a graph $\Gamma = (X, \sim)$ is defined by

$$(A)_{ij} = \begin{cases} 1 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

Here A is indexed by $(i, j) \in X \times X$. Notice that an adjacency matrix A only depends on the ordering of the vertices of the graph Γ, so we can abuse language and denote A as *the* adjacency matrix of Γ.

As we shall see in the following sections the adjacency matrix of a graph is a very important tool to investigate the properties of a graph. Vice versa, for a given symmetric 0-1-*matrix* $A \in \mathbb{C}^{n \times n}$ (i.e. A is symmetric and all entries of A are in $\{0, 1\}$) we can define a graph $\Gamma = (X, \sim)$ if we take the *canonical basis* $\{e_1, \dots, e_n\}$ of $\mathbb{C}^{n \times n}$ as X and define $\sim$ by

$$e_i \sim e_j \text{ if and only if } e_i^T A e_j = 1.$$

Hence, we can identify a graph with its adjacency matrix. For example the phrase "Let λ_- be the smallest eigenvalue of the graph Γ." means the smallest eigenvalues of the adjacency matrix of Γ.

There exists a more general concept of an adjacency matrix, the *extended weight adjacency matrix* of a graph $\Gamma = (X, \sim)$. We say that a symmetric matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is an extended weight adjacency matrix of Γ if

(a) $\mathbb{R} \ni a_{ij} \leqslant 0$ if i and j are non-adjacent,

(b) $a_{ii} = 0$,

(c) the all-ones vector $\mathbf{j}$ is an eigenvector of A.

(d) $a_{ij} > 0$ for one pair (i, j) with i and j adjacent.

We denote the eigenvalue of $\mathbf{j}$ by k. Obviously, the (0-1-)adjacency matrix of a k-regular graph is an extended weight adjacency matrix. In the notation of Section 1.1 we denote the eigenspaces of a k-regular extended weighted adjacency matrix A by $V_0, \dots, V_d$, their associated eigenvalues by $\lambda_0, \dots, \lambda_d$, and their multiplicities by $f_0, \dots, f_d$. We suppose $\lambda_0 = k$ and $\mathbf{j} \in V_0$. Note that the eigenspace V of the eigenvector $\mathbf{j}$ might have a dimension larger than 1. In this case we find a subspace $V_{d+1} \subset V$ with $V = \langle \mathbf{j} \rangle \perp V_{d+1}$ and $\dim(V_{d+1}) = f_0 - 1$. Additionally, we can easily define an orthonormal projection matrix E_{d+1} onto V_{d+1}. Therefore we can assume without loss of generality $V_0 = \langle \mathbf{j} \rangle$, $f_0 = 1$ and

$$E_0 = n^{-1} J \tag{1.3}$$

throughout this thesis. As this simplifies many statements significantly, we shall do so. Alternatively, one could assume that all considered graphs are *connected* (i.e. no two vertices x, y of the graph satisfy $d(x, y) = \infty$). As one major class of graphs considered in this thesis, the dual polar graphs of the hyperbolic quadric $Q^+(2d-1, q)$, d even, are not connected, we prefer to proceed with this assumption on V_0.

1.3 ASSOCIATION SCHEMES

Association schemes generalize distance-regular graphs. The exact definition of an association scheme varies.[2] We use the definition of [13, Ch. 2]. In this work an association scheme is what others might call a *symmetric association scheme*. It is defined as follows.

Definition 1.4. *Let* X *be a finite set. A* d*-class* association scheme *is a pair* $(X, \mathcal{R})$*, where* $\mathcal{R} = \{R_0, \dots, R_d\}$ *is a set of symmetric binary relations on* X *with the following properties:*

(a) R_0 *is the identity relation.*

(b) $\mathcal{R}$ *is a partition of* $X \times X$.

(c) There are numbers p_{ij}^k *such that for* $x, y \in X$ *with* xR_ky *there are exactly* p_{ij}^k *elements* z *with* xR_iz *and* zR_jy.

2 See the following blog entry by Peter Cameron for an exhaustive discussion of this topic: http://cameroncounts.wordpress.com/2014/06/08/terminology-association-scheme-or-coherent-configuration/ (retrieved: 26/11/2014)

Remark 1.5. *It is a common practice to shorten association scheme to* scheme*. This scheme is unrelated to the schemes known from algebraic geometry.*

The number $n_i := p^0_{ii}$ is called the i-*valency* of $(X, \mathcal{R})$. The total number of elements of X is

$$n := |X| = \sum_{i=0}^{d} n_i.$$

The relations R_i are described by their *adjacency matrices* $A_i \in \mathbb{C}^{n\times n}$ defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } xR_iy \\ 0 & \text{otherwise.} \end{cases}$$

The matrices A_i are symmetric 0-1-matrices, so we can identify them with a graph with the vertex set X as seen in Section 1.2. We shall call this graph as the *graph of* R_i. We can use these matrices to restate Definition 1.4. Then symmetric 0-1-matrices $A_0, \dots, A_d \in \mathbb{C}^{n\times n}$ belong to a d-class association scheme if

(a) $A_0 = I$,

(b) $\sum_{i=0}^{d} A_i = J$,

(c) there exist nonnegative integers p^k_{ij} such that

$$A_iA_j = \sum_{k=0}^{d} p^k_{ij} A_k$$

for all $(i, j, k) \in \{0, \dots, d\} \times \{0, \dots, d\} \times \{0, \dots, d\}$.

The symmetric matrices A_i are linearly independent by (b), so they generate a $(d + 1)$-dimensional algebra by (c). This algebra is called the *Bose-Mesner algebra* of the scheme.

The matrices A_i are symmetric, so they have n real eigenvalues by Lemma 1.1. Additionally, they commute, so we can diagonalize them simultaneously [33, 4.3.6, p. 239]. Therefore, we can decompose

$\mathbb{C}^n$ into the $d+1$ common eigenspaces $V_0, \dots, V_d$ of $A_0, \dots, A_d$. We denote the dimension of V_i by f_i. We have that $J = \sum_{i=0}^{d} A_i$, so the eigenspaces $V_0, \dots, V_d$ are eigenspaces of J. The matrix J has n as an eigenvalue with multiplicity 1, and 0 as an eigenvalue with multiplicity $n-1$. Therefore, we have $f_i = 1$ for one i. We suppose $f_0 = 1$.

Notice that our E_is here are the same as in Theorem 1.3. Hence, we can use Theorem 1.3 and the preceding notes. In particular, all E_i are idempotents, their sum is the identity matrix, and $E_0 = n^{-1}J$.

We define the $(d+1) \times (d+1)$-matrices P and Q over $\mathbb{C}$ by

$$A_j = \sum_{i=0}^{d} P_{ij} E_i, \qquad\qquad E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i.$$

The matrices P and Q are called the *eigenvalue matrices* of the association scheme. Notice that some authors define Q as $E_j = \sum_{i=0}^{d} Q_{ij} A_i$. By the pairwise orthogonality of the E_is, we have

$$A_j E_i = \left(\sum_{i=0}^{d} P_{ij} E_i \right) E_i = P_{ij} E_i.$$

This shows that the P_{ij}s are exactly the eigenvalues of the A_js. Also notice that the linear independence of the E_is and

$$\begin{aligned} E_j &= \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i \\ &= \frac{1}{n} \sum_{i=0}^{d} Q_{ij} \sum_{k=0}^{d} P_{ki} E_k \\ &= \frac{1}{n} \sum_{i=0}^{d} Q_{ij} P_{ji} E_j \end{aligned}$$

imply that $PQ = nI = QP$.

We shall summarize some well-known properties of the eigenvalue matrices in the following.

Lemma 1.6 ([13, Lemma 2.2.1])**.** *The eigenmatrices* P *and* Q *of an association scheme satisfy the following.*

(a) $P_{i0} = Q_{i0} = 1$,

(b) $P_{0j} = n_j$, $Q_{0j} = f_j$,

(c) $f_i P_{ij} = n_j Q_{ji}$.

Association schemes are often used to investigate distance-regular graphs $\Gamma = (X, \sim)$ of diameter d. The graph Γ can be seen as an association scheme if we define the relations $R_0, \ldots, R_d$ by

$$xR_i y = \begin{cases} 1 & \text{if } d(x,y) = i, \\ 0 & \text{otherwise.} \end{cases}$$

There are many other general mathematical structures that can be considered as association schemes, e.g. groups and buildings provide many examples for schemes [81, Introduction].

Remark 1.7. *The reader might wonder if there exists a natural ordering of the A_is and E_is.*

(a) *So-called* metric *(or* P-polynomial*) schemes have a natural ordering of the matrices A_i. A scheme with an ordering of the matrices A_i is metric if $p_{ij}^k \neq 0$ implies $k \leqslant i + j$ and moreover $p_{ij}^{i+j} \neq 0$ for all i, j, k.*

(b) *So-called* cometric *(or* Q-polynomial*) schemes have a natural ordering of the matrices E_i. The Bose-Mesner algebra is closed under componentwise (bad student's, Hadamard, Schur) multiplication $\circ$. Hence we can define the* Krein parameters *or* dual intersection numbers q_{ij}^k *by*

$$E_i \circ E_j = \frac{1}{n} \sum_{k=0}^{d} q_{ij}^k E_k.$$

A scheme is cometric if the q_{ij}^k satisfy the analog conditions as the p_{ij}^k.

All relevant association schemes in this work are metric and cometric. We will always use the A_i, E_i, R_i, and V_i in the ordering used by the cited source. This is always a metric and cometric ordering.

Also notice that these orderings are not unique. It is an easy exercise to see that (A_0, A_1, A_2) and (A_0, A_2, A_1) are both metric orderings if A_1 is a connected strongly-regular graph.

1.4 ALGEBRAIC COMBINATORICS

The purpose of this chapter is to introduce some important concepts of algebraic combinatorics and algebraic graph theory. The main idea of algebraic graph theory is to take a graph Γ and to associate it with an algebraic object such as the eigenvalues of its adjacency matrix. Then the properties of this object provide new insights on X itself. We restrict ourselves to the adjacency matrices of our graphs and association schemes. While we only work in distance-regular graphs, we keep the results as general as it seems appropriate.

We want to investigate subsets of a graph $\Gamma = (X, \sim)$. The basic tool for the investigation of subsets Y of X is the characteristic vector χ_Y indexed by X which is defined by

$$(\chi_Y)_x = \begin{cases} 1 & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y. \end{cases}$$

1.4.1 *Regular Graphs*

In this section $\Gamma = (X, \sim)$ denotes a regular simple graph with n vertices. The matrix A is an (extended weight) adjacency matrix of Γ with eigenvalues $\lambda_0 = k, \lambda_1 \ldots, \lambda_d$. Suppose that k is the eigenvalue of $\mathbf{j}$. We denote the smallest eigenvalue of A by λ_-, the associated eigenspace by V_-, the associated idempotent matrix by E_-, and its multiplicity by f_-. As usual, we denote the multiplicity of λ_i by f_i.

Recall that we always assume without loss of generality $E_0 = \frac{1}{n}J$ and that we can write χ_Y as a sum of orthogonal eigenvectors as follows.

$$\begin{aligned}\chi_Y &= \sum_{i=0}^{d} E_i \chi_Y \\ &= \frac{1}{n} J\chi_Y + \sum_{i=1}^{d} E_i \chi_Y \\ &= \frac{|Y|}{n}\mathbf{j} + \sum_{i=1}^{d} E_i \chi_Y.\end{aligned}$$

Hence, we have the following Lemma.

Lemma 1.8. *Let Y be a subset of X. Then*

$$\chi_Y = \frac{|Y|}{n}\mathbf{j} + \sum_{i=1}^{d} E_i \chi_Y.$$

The following result is well-known, but we include a proof as it is a good introduction to the proof of Proposition 1.12.

Proposition 1.9 (Hoffman's Bound for Independent Sets)**.** *Let $Y \subseteq X$ be an independent set of Γ and $k + \lambda_- > 0$. Then*

$$|Y| \leqslant \frac{n\lambda_-}{\lambda_- - k}$$

with equality only if $\chi_Y \in \langle \mathbf{j} \rangle \perp V_-$.

Proof. Recall that, by Theorem 1.3, we can decompose A into pairwise orthogonal idempotent matrices E_i as

$$A = \frac{k}{n}J + \sum_{i=1}^{d} \lambda_i E_i,$$

Let χ be χ_Y, the characteristic vector of Y. We have that Y is an independent set and A an extended weight adjacency matrix, so $\chi^T A \chi \leqslant 0$. We put $y = |Y|$. By Lemma 1.8,

$$\chi = \frac{y}{n}\mathbf{j} + \sum_{i=1}^{d} E_i \chi.$$

By $E_i E_j = \delta_{ij} E_i^2$, we have

$$\begin{aligned}
0 &\geqslant \chi^T A \chi \\
&= \frac{k}{n}\chi^T J \chi + \sum_{i=1}^{d} \lambda_i \chi^T E_i \chi \\
&= \frac{ky^2}{n} + \sum_{i=1}^{d} \lambda_i (E_i\chi)^T (E_i\chi) \\
&\geqslant \frac{ky^2}{n} + \lambda_- \sum_{i=1}^{d} (E_i\chi)^T (E_i\chi) \qquad (1.4)
\end{aligned}$$

Furthermore, χ is a 0-1-vector, so

$$y = \chi^T \chi = \frac{y^2}{n^2}\mathbf{j}^T\mathbf{j} + \sum_{i=1}^{d} (E_i\chi)^T (E_i\chi). \qquad (1.5)$$

If we put this back into our previous inequality (1.4), then we have

$$\begin{aligned}
0 &\geqslant \frac{ky^2}{n} + \sum_{i=1}^{d} \lambda_i (y - \frac{y^2}{n}) \\
&\geqslant \frac{ky^2}{n} + \lambda_- \left(y - \frac{y^2}{n}\right). \qquad (1.6)
\end{aligned}$$

Rearranging yields the first part of the assertion.

If this bound is tight, then we have equality in (1.4) and (1.6). Hence,

$$\chi = \frac{y}{n}\mathbf{j} + E_-\chi$$

which shows the second part of the assertion. □

Remark 1.10. *Am short remark on the name of Hoffman's bound. The previous result,* Hoffman's bound, *is (with* 0-1*-adjacency matrices) due to an unpublished result by Hoffman. It was independently discovered for the special case of association schemes (and published) by Delsarte, so some call it* Delsarte bound, *some call it* Hoffman-Delsarte bound. *Others call it* ratio bound. *All these names are sometimes accompanied by adjectives such as* weighted *or* generalized, *since original statements of the bound did restrict themselves to* 0-1*-adjacency matrices.*

There are plenty generalizations of Hoffman's bound. For reasons which will be explained in Chapter 6 the author found some interest in the following variant of the problem: Instead of a set of pairwise non-adjacent vertices Y of Γ consider a pair (Y, Z) with $Y, Z \subseteq X$ such that all vertices of Y are non-adjacent to all vertices of Z. We say that such a pair (Y, Z) is a *cross-intersecting independent set*.[3]

Let λ_+ be the largest eigenvalue besides k (with $\lambda_+ = k$ if the multiplicity of k is larger than 1 and k is the largest eigenvalue). Again, denote the associated idempotent, eigenspace, and multiplicity by E_+, V_+, and f_+. Define $\lambda_b = \max\{\lambda_+, -\lambda_-\}$. We say that λ_b is the *second largest absolute eigenvalue* of Γ. Hoffman's bound for cross-intersecting sets seems to be first published by Willem H. Haemers in [40]. The author learned about this technique from a paper by Tokushige [75] where he uses a variant of the result based on the work of Ellis, Friedgut, and Pipel [31]. The proof is nearly identical to the proof of the normal Hoffman bound.

We have to restate one of the proofs, since all the mentioned publications did not characterize the case of tightness in a way that is specific enough for our application of this result.

Lemma 1.11 (Inequality of Arithmetic and Geometric Means)**.** *Let* $0 \leqslant \alpha, \beta \leqslant 1$. *Then we have*

$$\sqrt{1-\alpha}\sqrt{1-\beta} \leqslant 1 - \sqrt{\alpha\beta}$$

with equality if and only if $\alpha = \beta$.

Proposition 1.12 (Hoffman's Bound for Cross-Intersecting Independent Sets)**.** *Let* (Y, Z) *be a cross-intersecting independent set of* Γ *and* $k + \lambda_b > 0$. *Then*

$$\sqrt{|Y| \cdot |Z|} \leqslant \frac{n\lambda_b}{k + \lambda_b}.$$

If equality holds, then $|Y| = |Z|$. *Define* α *by* $|Y| = |Z| = \alpha n$. *Furthermore, one of the following cases occurs.*

(a) We have $\lambda_+ = \lambda_b > -\lambda_-$, $\chi_Y = \alpha\mathbf{j} + v_+$, *and* $\chi_Z = \alpha\mathbf{j} - v_+$ *for some vector* $v_+ \in V_+$.

3 This name is non-standard.

(b) *We have* $\lambda_+ < \lambda_b = -\lambda_-$, $\chi_Y = \alpha\mathbf{j} + \nu_-$, *and* $\chi_Z = \alpha\mathbf{j} + \nu_-$ *for some vector* $\nu_- \in V_-$. *In this case* $Y = Z$, *and* Y *is an independent set of maximum size.*

(c) *We have* $\lambda_+ = \lambda_b = -\lambda_-$, $\chi_Y = \alpha\mathbf{j} + \nu_- + \nu_+$, *and* $\chi_Z = \alpha\mathbf{j} + \nu_- - \nu_+$ *for some vectors* $\nu_- \in V_-$ *and* $\nu_+ \in V_+$.

Proof. Let χ be χ_Y, the characteristic vector of Y. Let ψ be χ_Z, the characteristic vector of Z. We have that (Y, Z) is a cross-intersecting independent set and A an extended weight adjacency matrix, so $\chi^T A\psi \leqslant 0$

Put $y = |Y|$ and $z = |Z|$. Suppose without loss of generality $y \geqslant z > 0$. By Lemma 1.8,

$$\chi = \frac{y}{n}\mathbf{j} + \sum_{i=1}^{d} E_i\chi,$$

$$\psi = \frac{z}{n}\mathbf{j} + \sum_{i=1}^{d} E_i\psi.$$

Similar to the proof of Proposition 1.9, we have

$$\begin{aligned}
0 &\geqslant \chi^T A\psi \\
&= \frac{k}{n}\chi^T J\psi - \left|\sum_{i=1}^{d} \lambda_i \chi^T E_i \psi\right| && (1.7)\\
&\geqslant \frac{kyz}{n} - \sum_{i=1}^{d} |\lambda_i (E_i\chi)^T (E_i\psi)| && (1.8)\\
&\geqslant \frac{kyz}{n} - \lambda_b \sum_{i=1}^{d} |E_i\chi| \cdot |E_i\psi| && (1.9)\\
&\geqslant \frac{kyz}{n} - \lambda_b \sqrt{\sum_{i=1}^{d} (E_i\chi)^T (E_i\chi)} \cdot \sqrt{\sum_{i=1}^{d} (E_i\psi)^T (E_i\psi)} && (1.10)
\end{aligned}$$

In (1.7) equality holds $\frac{k}{n}\chi^T J \psi > 0$, so the remaining term is negative. Notice that we apply the Cauchy-Schwartz inequality for (1.9) and (1.10). Furthermore, as in (1.4),

$$y = \frac{y^2}{n} + \sum_{i=1}^{d} (E_i\chi)^T(E_i\chi), \tag{1.11}$$
$$z = \frac{z^2}{n} + \sum_{i=1}^{d} (E_i\psi)^T(E_i\psi).$$

If we put this back into our previous inequality (1.10), then we have by Lemma 1.11 and similar to the proof of Lemma 1.9

$$0 \geqslant \frac{kyz}{n} - \lambda_b \sqrt{yz}\sqrt{1-\frac{y}{n}}\sqrt{1-\frac{z}{n}} \tag{1.12}$$
$$\geqslant \frac{kyz}{n} - \lambda_b\sqrt{yz}(1-\sqrt{yz}/n). \tag{1.13}$$

Rearranging yields the first part of the assertion.

If this bound is tight, then we have equality from (1.7) to (1.13). In particular, equality in Lemma 1.11 and (1.13) shows $y = z$. Equality in (1.9) shows

$$\chi = \frac{y}{n}\mathbf{j} + E_-\chi + E_+\chi$$
$$\psi = \frac{z}{n}\mathbf{j} + E_-\psi + E_+\psi.$$

Equality in (1.7) shows $E_-\chi = E_-\psi$ and $E_+\chi = -E_+\psi$. □

1.4.2 *Association Schemes*

This subsection presents those results from algebraic combinatorics which are (in the context of this thesis) best stated in the context of association schemes. We will mostly consider the following type of problems. Let $(X, \mathcal{R})$ be a d-class association scheme with the usual notation (see Section 1.3). Let $I \subseteq \{1, \dots, d\}$. Let Y be a subset of X such that all pairwise different elements $y, y' \in Y$ satisfy yR_iy' for some $i \in I$. What are the properties of Y? In particular, can we bound the possible sizes of Y and can we describe Y if it reaches one of these bounds?

In the following we adapt the notation of [13, Sec. 2.5]. Let Y be a nonempty subset of X. Let χ be χ_Y, the characteristic vector of Y. Then we define $a = (a_0, a_1, \ldots, a_d)$, the *inner distribution* of Y, by

$$a_i = \frac{1}{|Y|}\chi^T A_i \chi.$$

This means that a_i is the average number of intersections in Y in relation R_i. Alternatively,

$$a_i = \frac{1}{|Y|}|\{(y, y') \in Y \times Y : yR_i y'\}|.$$

Notice that $a_0 = 1$ and $\sum_{i=0}^{d} a_d = |Y|$.

For $I \subseteq \{1, \ldots, d\}$ we call Y an I-*clique* if its inner distribution a satisfies $a_i = 0$ for all $i \in \{1, \ldots, d\} \setminus I$. We call Y an I-*coclique* if its inner distribution a satisfies $a_i = 0$ for all $i \in I$. Let $\bar{I}$ denote $\{1, \ldots, d\} \setminus I$. Some trivial observations:

(a) An I-clique is an $\bar{I}$-coclique and vice versa.

(b) An I-clique is a clique of the graph with adjacency matrix $\sum_{i \in I} A_i$. An I-clique is an independent set of the graph with adjacency matrix $\sum_{i \in \bar{I}} A_i$.

(c) Graphs with an adjacency matrix of the form $\sum_{i \in I} A_i$ are regular.

In particular, we already have the tools presented in 1.4.1 at our disposal to investigate I-cliques and I-cocliques.

Lemma 1.13 (Delsarte [25]). *The inner distribution* a *of a nonempty set* $Y \subseteq X$ *satisfies*

$$(|Y|aQ)_i = n\chi^T E_i \chi.$$

Proof. By the definitions of Q and a, we have

$$\begin{aligned}(|Y|aQ)_i = |Y| \sum_{k=0}^{d} a_k Q_{ki} &= \sum_{k=0}^{d} \chi^T A_k \chi Q_{ki} \\ &= \chi^T \left(\sum_{k=0}^{d} Q_{ki} A_k \right) \chi = n\chi^T E_i \chi.\end{aligned}$$

□

The matrices E_i are positive semidefinite, so Lemma 1.13 shows the following result, *Delsarte's linear programming bound*.

Proposition 1.14 (Delsarte [25]). *The inner distribution vector* a *of a nonempty set* $Y \subseteq X$ *satisfies* $aQ \geqslant 0$. *If* $(aQ)_i = 0$, *then* $\chi_Y \in V_i^\perp$.

This proposition defines linear constraints on the inner distribution of any subset Y of X. E.g. if we consider $\{1\}$-cliques with $d = 3$, then we have $a = (1, a_1, 0, 0)$. Hence, we have the linear constraints $Q_{0i} + a_1 Q_{1i} \geqslant 0$, $i \in \{1, 2, 3\}$, and we want to maximize the non-negative variable a_1. This is a linear programming problem with 4 constraints and 1 variable. This problem is bounded, since we have $Q_{1i} = f_i P_{i1}/n_1 < 0$ for at least one i. One can use standard algorithms to solve this for given parameters efficiently (see any book on linear optimization such as [64]).

It was shown by Luz [59] (and appears to have been folklore even before his publication) that Proposition 1.14 is a special case of Proposition 1.9. Normally, the same holds for generalizations of Hoffman's bound such as Proposition 1.12. Again, all the different names mentioned in Remark 1.10 seem to be in use for this bound.

There exists a bound for cliques which is also called Hoffman's bound. The following version is due to Chris Godsil who contributed the multiplicity argument. We restate the proof as Godsil did formulate the result for cliques of regular graphs, while we focus on association schemes here.

Proposition 1.15 (Godsil [36, Theorem 3.5]). *Let* Y *be a subset of* X *under the condition that all elements of* Y *are pairwise in relation* R_0 *or* R_i. *Let* $j \in \{1, \ldots, d\}$.

(a) If $P_{ji} < 0$, *then we have*

$$|Y| \leqslant 1 - n_i/P_{ji}$$

with equality only if $\chi_Y \in V_j^\perp$.

(b) If $Q_{0j} \neq Q_{ij}$, *then we have*

$$|Y| \leqslant 1 + f_j$$

with equality only if $|Y| = 1 - n_i/P_{ji}$, *equivalently,* $Q_{ij} \neq -1$.

Proof. Recall

$$nE_j = \sum_{i=0}^{d} Q_{ij} A_i.$$

The entry $(E_j)_{xy}$ only depends on the k for which we have xR_ky. Therefore, the $|Y| \times |Y|$-submatrix S of E_j indexed by Y has the form $\alpha I + \beta J$. Let χ be χ_Y, the characteristic vector of Y and let $\chi_{|Y}$ be the restriction of χ to the vertices in Y. Then we have

$$0 \leqslant \chi^T E_j \chi = \chi_{|Y}^T (\alpha I + \beta J) \chi_{|Y} = \alpha |Y| + \beta |Y|^2 \tag{1.14}$$

with equality only if $\chi_Y \in V_j^{\perp}$. By the definition of E_j and Lemma 1.6, we have

$$n\alpha = Q_{0j} - Q_{ij} = f_j (1 - \frac{P_{ji}}{n_i})$$
$$n\beta = Q_{ij} = \frac{f_j}{n_i} P_{ji}.$$

Rearranging (1.14) yields the first part of the assertion.

Furthermore,

$$\operatorname{rank}(S) = \operatorname{rank}(\alpha I + \beta J) \leqslant \operatorname{rank}(E_j) = f_j.$$

Now for $\alpha, \beta \neq 0$,

$$\operatorname{rank}(S) = |Y| - 1 \qquad \text{if } \alpha = -\beta |Y|,$$
$$\operatorname{rank}(S) = |Y| \qquad \text{if } \alpha \neq -\beta |Y|.$$

As $|Y| = 1 - n_i/P_{ji}$, we have that $|Y| = f_j + 1$ implies $-\frac{n_i}{P_{ji}} = f_j$. This shows the second part of the assertion. □

1.5 GRAPHS RELATED TO FINITE GEOMETRIES

The purpose of this chapter is to introduce some geometrical objects and their associated association schemes. See other sources such as [44, 45] for details. We adopt the ordering of the A_is and E_is from our sources.

1.5.1 *Sets*

Many problems in finite geometry come from analog problems on sets. Let $N = \{1, \dots, n\}$. The *Johnson graph* Γ of the k-sets of N has all k-element subsets of N as its vertices X. Two vertices x and y are adjacent if $|x \cap y| = k - 1$. We write $J(n, k)$ for Γ.

Theorem 1.16 ([13, Th. 9.1.2]). *Let Γ be $J(n, k)$. Then Γ has diameter $d = \min(k, n-k)$ and is distance-regular. The eigenvalues and multiplicities of Γ are given by*

$$\lambda_i = (k-i)(n-k-i) - i, \qquad f_i = \binom{n}{i} - \binom{n}{i-1}$$

for $i \in \{0, 1, \dots, d\}$.

1.5.2 *Vector Spaces*

Let V be an n-dimensional vector space of a finite field K of order q. The *Grassmann graph* Γ of the k-subspaces of V has the k-dimensional subspaces of V as its vertices. Two vertices x and y are adjacent if $\dim(x \cap y) = k - 1$. We write $J_q(n, k)$ for Γ.

For integers n, k the Gaussian coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \prod_{i=1}^{k} \frac{q^{n-i+1}-1}{q^i-1} & \text{if } 0 \leqslant k \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

We will write $\begin{bmatrix} n \\ k \end{bmatrix}$ instead of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ if the choice for q is clear from the context. An easy calculation shows

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}, \tag{1.15}$$

and for $(n, k) \neq (0, 0)$

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix} &= \begin{bmatrix} n-1 \\ k \end{bmatrix} q^k + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\ &= \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} q^{n-k}. \end{aligned} \tag{1.16}$$

A standard double counting argument shows that the number k-dimensional subspaces of an n-dimensional vector space over a finite field of order q is $\begin{bmatrix} n \\ k \end{bmatrix}$. We write $[n]$ for $\begin{bmatrix} n \\ 1 \end{bmatrix}$, the number of 1-dimensional subspaces of V. If P is a 1-dimensional subspace, then the factor space V/P contains $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ $(k-1)$-dimensional subspaces. Hence, exactly $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ k-dimensional subspaces of V contain a fixed 1-dimensional subspace. We shall investigate the Gaussian coefficient in some more detail in Section 2.

Theorem 1.17 ([13, Th. 9.3.3]). *Let Γ be $J_q(n, k)$. Then Γ has diameter $d = \min(k, n-k)$ and is distance-regular. The eigenvalues and multiplicities of Γ are given by*

$$\lambda_i = q^{i+1}[k-i][n-k-i], \qquad\qquad f_i = \begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n \\ i-1 \end{bmatrix}$$

for $i \in \{0, 1, \ldots, d\}$.

Notice that we can identify V with the *projective space* $\mathcal{P}$ of dimension $n-1$ over K where the 1-dimensional subspaces of V are the *points* of $\mathcal{P}$, and the 2-dimensional subspaces of V are the *lines* of $\mathcal{P}$. Furthermore, we call the $(n-1)$-dimensional subspaces of V *hyperplanes*.

A Grassmann graph is distance-regular, so we can easily define an association scheme on it by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \operatorname{codim}(x \cap y) = i, \\ 0 & \text{if } \operatorname{codim}(x \cap y) \neq i. \end{cases}$$

Here we define $\operatorname{codim}(x \cap y) = k - \dim(x \cap y)$. Formulas for the eigenvalues of these association schemes were calculated by Delsarte [26] and Eisfeld [30]. We use the notation of Vanhove [78, Theorem 3.2.4, Remark 3.2.5] whose PhD thesis is an excellent reference for the properties of Grassmann schemes (as well as the association scheme which belongs to finite classical polar spaces). In particular, Theorem 3.2.2 and Remark 3.2.5 in Section 3.2 of [78] show the following theorem.

Theorem 1.18 ([30]). *Let Y be the set of all k-dimensional spaces on a fixed point of $\mathbb{F}_q^n$. Then χ_Y is an eigenvector of A_i with associated eigenvalue*

$$\begin{bmatrix} n-k-1 \\ i \end{bmatrix} \begin{bmatrix} k-1 \\ i \end{bmatrix} q^{(i+1)i} - \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k-1 \\ i-1 \end{bmatrix} q^{i(i-1)}.$$

Proof. By [78, Theorem 3.2.2], the vector χ_Y is an eigenvector in the eigenspace V_1^k. By [78, Remark 3.2.5], (1.15), and (1.16), the corresponding eigenvalue is

$$\begin{aligned}
&\begin{bmatrix} n-k \\ n-k-i \end{bmatrix} \begin{bmatrix} k-1 \\ i \end{bmatrix} q^{i^2} - \begin{bmatrix} n-k-1 \\ n-k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} q^{i(i-1)} \\
&= \begin{bmatrix} n-k \\ i \end{bmatrix} \begin{bmatrix} k-1 \\ i \end{bmatrix} q^{i^2} - \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} q^{i(i-1)} \\
&= \begin{bmatrix} n-k-1 \\ i \end{bmatrix} \begin{bmatrix} k-1 \\ i \end{bmatrix} q^{(i+1)i} + \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k-1 \\ i \end{bmatrix} q^{i^2} \\
&\qquad\qquad - \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} q^{i(i-1)} \\
&= \begin{bmatrix} n-k-1 \\ i \end{bmatrix} \begin{bmatrix} k-1 \\ i \end{bmatrix} q^{(i+1)i} - \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k-1 \\ i-1 \end{bmatrix} q^{i(i-1)}.
\end{aligned}$$

□

To the knowledge of the author the property that the χ_Y of the previous result is an eigenvector of A_i was first explicitly observed by Frankl and Wilson [35]. It is not hard to notice this property while calculating the eigenvalues P_{ij}, so it should have been known to Delsarte as well.

1.5.3 *Finite Classical Polar Spaces*

A polar space is an incidence geometry introduced by Veldkamp [79]. We refer to Veldkamp [79] and Tits [74] for the theory of polar spaces. We refer to Hirschfeld [44] for an overview over Galois geometries.

Let $\mathbb{F}$ be a field. Let σ be a field automorphism of $\mathbb{F}$. Let V be a vector space over $\mathbb{F}$. A *sesquilinear form* f is a map $f : V \times V \to \mathbb{F}$ that is linear in its first argument and semilinear in its second argument, i.e. all $v, w, u \in V$ and all $a \in \mathbb{F}$ satisfy

- $f(v+w,u) = f(v,u) + f(w,u)$,
- $f(av,w) = af(v,w)$,
- $f(v,w+u) = f(v,w) + f(v,u)$,
- $f(v,aw) = a^{\sigma} f(v,w)$.

A sesquilinear form is called *reflexive form* if $f(v,w) = 0$ implies $f(w,v) = 0$ for all $v,w \in V$. A sesquilinear form is called *symplectic form* if $f(v,v) = 0$ for all $v \in V$. A sesquilinear form is called *Hermitian form* if $\sigma \neq \sigma^2 = \mathrm{id}$ and $f(v,w) = f(w,v)^{\sigma}$ for all $v,w \in V$.

A *quadratic form* Q is a map $Q : V \to \mathbb{F}$ such that

- $Q(av) = a^2 Q(v)$ for all $v \in V$ and $a \in \mathbb{F}$,
- there exists a sesquilinear form f with $\sigma = \mathrm{id}$ such that $Q(v + w) = Q(v) + Q(w) + f(v,w)$ for all $v,w \in V$.

We say that f is the associated bilinear form of Q.

A reflexive sesquilinear form f is called *degenerate* if there exists $v \in V \setminus \{0\}$ with $f(v,w) = 0$ for all $w \in V$. A quadratic form is called *degenerate* if there exists a vector $v \in V$ with $Q(v) = 0$ and $f(v,w) = 0$ for all $w \in V$. A reflexive sesquilinear form, respectively, quadratic form is called *non-degenerate* if it is not degenerate. A subspace is called *totally isotropic*, respectively, *totally singular* whenever the form vanishes completely on this subspace.

The set of totally isotropic subspaces of V with respect to a reflexive sesquilinear form f, respectively, quadratic form Q form a so-called *incidence geometry* with respect to f, respectively, Q. Two totally isotropic subspaces S and T are called *incident* if $S \subseteq T$ or $T \subseteq S$. If V is finite and f, respectively, Q is non-degenerate and reflexive, then the geometry defined by f, respectively, Q is a *polar space*. The maximal totally isotropic subspaces of a polar space are called *generators*. If the (vector space) dimension of a generator is larger than 2, then all finite polar spaces are *classical* [74]. The dimension of a generator is the same for all generators of a polar space. This dimension is called the *rank* of the polar space.

There exist the following types of finite classical polar spaces of rank d over a finite field of order q.

(a) The hyperbolic quadric $Q^+(2d-1,q)$. Up to coordinate transformation its non-degenerate quadratic form is $x_0x_1+\ldots+x_{2d-1}x_{2d}$.

(b) The parabolic quadric $Q(2d,q)$. Up to coordinate transformation its non-degenerate quadratic form is $x_0^2+x_1x_2+\ldots+x_{2d}x_{2d+1}$.

(c) The elliptic quadric $Q^-(2d+1,q)$. Up to coordinate transformation its non-degenerate quadratic form is $f(x_0,x_1)+x_2x_3+\ldots+x_{2d+1}x_{2d+2}$. Here $f(x_0,x_1)$ is an irreducible homogenous quadratic polynomial of $\mathbb{F}_q$.

(d) The Hermitian polar spaces $H(2d-1,q)$ and $H(2d,q)$. Here $q=r^2$ for a prime power r. Up to coordinate transformation the corresponding non-degenerate sesquilinear form is $x_0y_0^r+\ldots+x_ny_n^r$ for $H(n,q)$.

(e) The symplectic polar space $W(2d-1,q)$. Up to coordinate transformation the corresponding non-degenerate sesquilinear form is $x_0y_1-x_1y_0+\ldots+x_{2d-1}y_{2d}-x_{2d}y_{2d-1}$.

Remark 1.19. *We have the following conventions.*

(a) Unless otherwise mentioned, we use vector space dimension and not projective dimension.

(b) Whenever we say totally isotropic, then we mean totally singular if the considered polar space is a quadric.

Define the parameter e of a finite classical polar space as follows. The second and third columns contain alternative names (see [13, p. 274]). In the following table r is the natural number defined by $q=r^2$.

Polar Space	Chevalley Group	Graph	e
$Q^+(2d-1,q)$	$D_d(q)$	$\Omega^+(2d,q)$	0
$H(2d-1,q)$	${}^2A_{2d-1}(r)$	$U(2d,r)$	1/2
$Q(2d,q)$	$B_d(q)$	$\Omega(2d+1,q)$	1
$W(2d-1,q)$	$C_d(q)$	$Sp(2d,q)$	1
$H(2d,q)$	${}^2A_{2d}(r)$	$U(2d+1,r)$	3/2
$Q^-(2d+1,q)$	${}^2D_{d+1}(q)$	$\Omega^-(2d+2,q)$	2

We say that a polar space over $\mathbb{F}_q$ has *type* e. A polar space of rank d and type e has exactly

$$\prod_{i=0}^{d-1}(q^{i+e}+1) \tag{1.17}$$

generators. A polar space possesses

$$(q^{d-1+e}+1)[d] \tag{1.18}$$

totally isotropic points.

Let U be a subspace of V with an associated form f. If f is a quadratic form, then f restricted to U is also a quadratic form. If f is a symplectic form, then f restricted to U is also a symplectic form. If f is a Hermitian form, then f restricted to U is also a Hermitian form.

In the following we list a few more well-known properties of finite classical polar spaces, which we shall use throughout this thesis without further reference.

Remark 1.20. *The set of generators of an hyperbolic quadric of rank* d *can be partitioned into* Latin *and* Greek *generators. These are two sets of the same size such that*

(a) Latin generators meet each other in even codimension,

(b) Greek generators meet each other in even codimension,

(c) Latin generators meet Greek generators in odd codimension.

Here the codimension of a subspace S *is defined as* $d-\dim(S)$.

Remark 1.21. *If* q *is even, then the polar spaces* $W(2d-1,q)$ *and* $Q(2d,q)$ *are isomorphic.*

Let f be a non-degenerate sesquilinear form of a vector space V of dimension n. Let U be a subspace of V. Define $U^\perp$, the perp of U, by

$$U^\perp = \{v \in V : f(v,w) = 0 \text{ for all } w \in U\}.$$

If f is non-degenerate, then $\perp$ maps c-spaces onto $(n-c)$-spaces. Particularly, f induces a bijection between points and hyperplanes of V. In this case f is an example for a *duality* on V.

The *dual polar graph* Γ has the generators of a polar space $\mathcal{P}$ as its vertices, and two vertices x, y are adjacent if $\dim(x \cap y) = d - 1$.

Theorem 1.22 ([13, Th. 9.4.3]). *Let Γ be the dual polar graph of a polar space of rank d and type e. Then Γ has diameter d and is distance-regular. The eigenvalues and multiplicities of Γ are given by*

$$\lambda_i = q^e[d-i] - [i],$$
$$f_i = q^i \begin{bmatrix} d \\ i \end{bmatrix} \frac{1 + q^{d+e-2i}}{1 + q^{d+e-i}} \prod_{j=1}^{i} \frac{1 + q^{d+e-j}}{1 + q^{j-e}}$$

for $i \in \{0, 1, \ldots, d\}$.

A dual polar graph is distance-regular, so we can easily define an association scheme on it by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \operatorname{codim}(x \cap y) = i, \\ 0 & \text{if } \operatorname{codim}(x \cap y) \neq i. \end{cases}$$

Formulas for the eigenvalues of these association schemes were calculated by Stanton [70], Eisfeld [30], and Vanhove [78, Theorem 4.3.6]. We will use Frédérics Vanhove's version which is stated in the following. We do explicitly state the eigenvalues of A_d as these are the most important for us.

Theorem 1.23. *For a polar space of rank d and type e, the eigenvalues of the adjacency matrix A_s are*

$$P_{r,s} = \sum_{t=\max(r-s,0)}^{\min(d-s,r)} (-1)^{r-t} \begin{bmatrix} d-r \\ d-s-t \end{bmatrix} \begin{bmatrix} r \\ t \end{bmatrix} q^{\binom{r-t}{2} + \binom{s-r+t}{2} + (s-r+t)e}.$$

Particularly, the eigenvalues of A_d are

$$P_{r,d} = (-1)^r q^{\binom{d-r}{2} + \binom{r}{2} + e(d-r)}. \tag{1.19}$$

1.6 COUNTING IN THESE GEOMETRIES

This section collects some of the necessary counting results which we shall need in the following chapters.

Recall that $P_{0s} = n_s$, so we can use Theorem 1.23 also as a reference for some basic counting arguments.[4] We will do some more calculations with these numbers in the individual chapters to obtain our specific results.

Corollary 1.24. *In a polar space of type* e *and rank* d*, exactly*

$$\begin{bmatrix} d \\ d-s \end{bmatrix} q^{\binom{s}{2}+se}.$$

generators meet a fixed generator in codimension s.

Definition 1.25. *Let* r, s, u, z, z_1, z_2, t *be integers. Let* $d \geqslant 0$. *Let*

- $\psi_{12}(d, r, s, u)$ *be the number of* r*-spaces meeting a fixed* s*-space in a fixed* u*-space in* $\mathbb{F}_q^d$ *if* $0 \leqslant u \leqslant s \leqslant d$, 0 *otherwise.*
- $\psi_2(d, r, s, u)$ *be the number of* r*-spaces meeting a fixed* s*-space in some* u*-space in* $\mathbb{F}_q^d$ *if* $0 \leqslant s \leqslant d$, 0 *otherwise.*
- $\psi_3(d, x, y, z, z_1, z_2)$ *be the number of* z*-spaces that meet a fixed* x*-space* X *in some* z_2*-space and a fixed* y*-space* $Y \subseteq X$ *in some* z_1*-space in* $\mathbb{F}_q^d$ *if* $0 \leqslant y \leqslant x \leqslant d$, 0 *otherwise.*

Theorem 1.26 ([44, Theorem 3.3, p. 88])**.** *Let* r, s, u, z, z_1, z_2, t *be integers. Let* $d \geqslant 0$. *We have the following identities.*

(a) $\psi_{12}(d, r, s, u) =$

$$q^{(r-u)(s-u)} \begin{bmatrix} d-s \\ r-u \end{bmatrix}.$$

(b) $\psi_2(d, r, s, u) =$

$$\begin{bmatrix} s \\ u \end{bmatrix} \psi_{12}(d, r, s, u).$$

4 Eisfeld, Stanton and Vanhove used n_s to obtain formulas for P_{0s} and not the other way around.

(c) $\psi_3(d, x, y, z, z_1, z_2) =$

$$q^{(z_2-z_1)(y-z_1)+(z-z_2)(x-z_2)} \begin{bmatrix} y \\ z_1 \end{bmatrix} \begin{bmatrix} x-y \\ z_2-z_1 \end{bmatrix} \begin{bmatrix} d-x \\ z-z_2 \end{bmatrix}.$$

Proof. By [45, Theorem 3.3, (1), p. 88],

$$\psi_{12}(d, r, s, u) = q^{(r-u)(s-u)} \begin{bmatrix} d-s \\ r-u \end{bmatrix}.$$

By [45, Theorem 3.3, (2), p. 88],

$$\psi_2(d, r, s, u) = \begin{bmatrix} s \\ u \end{bmatrix} \psi_{12}(d, r, s, u).$$

First we calculate $\psi_3(d, x, y, z, z_1, z_2)$. For this consider an x-dimensional subspace X and a y-dimensional subspace Y with $Y \subseteq X$. We want to choose a z-dimensional subspace Z with

$$\dim(Z \cap Y) = z_1, \qquad \dim(Z \cap X) = z_2.$$

The number of ways choosing $Z \cap X$ is $\psi_2(x, z_2, y, z_1)$. Then the number of ways choosing Z through a fixed subspace $Z \cap X$ is exactly $\psi_{12}(d, z, x, z_2)$. Hence, we have

$$\psi_3(d, x, y, z, z_1, z_2) = \psi_2(x, z_2, y, z_1)\psi_{12}(d, z, x, z_2).$$

□

Corollary 1.27. *Let $n \geqslant k, a \geqslant 1$. Let P be a point. Let S be an a-dimensional subspace of an n-dimensional vector space which does not contain P. Then there are*

$$q^{a(k-1)} \begin{bmatrix} n-a-1 \\ k-1 \end{bmatrix}$$

k-dimensional subspaces which do contain P and meet S trivially.

Proof. In the factor space of P we see that we have $\psi_{12}(n-1, k-1, a, 0)$ possibilities to choose the k-dimensional subspace. Theorem 1.26 shows the assertion. □

The Gaussian coefficient satisfies many interesting identities. One of them is the following.

Lemma 1.28. *Let* $n \neq 0$. *Let* $a \geqslant 0$. *Then,*

$$\sum_{k=0}^{a} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} = (-1)^{n+a} \begin{bmatrix} n-1 \\ a \end{bmatrix} q^{\binom{n-a}{2}}.$$

Proof.

$$\begin{aligned}
&\sum_{k=0}^{a} (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \\
\overset{(1.16)}{=} &\sum_{k=0}^{a} (-1)^{n-k} q^{\binom{n-k}{2}} \left(q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix} \right) \\
= &\sum_{k=0}^{a} (-1)^{n-k} \left(q^{\binom{n-k+1}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^{\binom{n-k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix} \right) \\
= &(-1)^{n-a} q^{\binom{n-a}{2}} \begin{bmatrix} n-1 \\ a \end{bmatrix}.
\end{aligned}$$

□

2 INEQUALITIES

The purpose of this chapter is to provide some useful inequalities for the Gaussian coefficient and the number of generators in a polar space.

2.1 BOUNDS ON THE GAUSSIAN COEFFICIENT

We start with several bounds on the Gaussian coefficient.

Lemma 2.1. *Let* $n \geqslant k \geqslant 0$.

(a) Let $q \geqslant 2$. *Then*

$$\begin{bmatrix} n \\ k \end{bmatrix} \leqslant \frac{111}{32} q^{k(n-k)} \leqslant \frac{7}{2} q^{k(n-k)}.$$

(b) Let $q \geqslant 3$. *Then*

$$\begin{bmatrix} n \\ k \end{bmatrix} \leqslant 2q^{k(n-k)}.$$

(c) Let $q \geqslant 4$. *Then*

$$\begin{bmatrix} n \\ k \end{bmatrix} \leqslant (1 + 2q^{-1}) q^{k(n-k)}.$$

(d) Let $q \geqslant 2$. *Let* $n \geqslant 1$. *Then*

$$[n] \leqslant \frac{q}{q-1} q^{n-1}.$$

Proof. First we prove the first three claims.

By definition,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{i=1}^{k} \frac{q^{n-k+i} - 1}{q^i - 1} \leqslant \prod_{i=1}^{k} \frac{q^{n-k+i}}{q^i - 1} = q^{k(n-k)} \prod_{i=1}^{k} \frac{q^i}{q^i - 1}. \tag{2.1}$$

Our claim is

$$\begin{bmatrix} n \\ k \end{bmatrix} \leqslant (1 + \alpha q^{-1}) q^{k(n-k)} \tag{2.2}$$

for $\alpha = \frac{79}{16}$ and $q = 2$, $\alpha = 3$ and $q = 3$, respectively, $\alpha = 2$ and $q \geqslant 4$.

The basis. In the following we check for $k \leqslant 8$ that

$$\prod_{i=1}^{k} \frac{q^i}{q^i - 1} \leqslant 1 + \alpha q^{-1},$$

so (2.2) holds by (2.1) for $k \leqslant 8$. For $q = 2, 3$ this is trivial. For $q \geqslant 4$ we that

$$\left(q(1 + 2q^{-1}) \prod_{i=1}^{k} (q^i - 1) \right) - q^{\binom{k+1}{2}}$$

evaluates to

$$\begin{aligned}
&2,\\
&q - 2,\\
&q^3 - 3q^2 - q + 2,\\
&q^6 - 3q^5 - 2q^4 + q^3 + 3q^2 + q - 2,\\
&q^{10} - 3q^9 - 2q^8 + 2q^6 + 4q^5 - q^3 - 3q^2 - q + 2,\\
&q^{15} - 3q^{14} - 2q^{13} + q^{11} + 3q^{10} + 3q^9\\
&\quad + q^8 - 3q^7 - 3q^6 - 2q^5 + q^3 + 3q^2 + q - 2,\\
&q^{21} - 3q^{20} - 2q^{19} + q^{17} + 2q^{16} + 2q^{15} + 4q^{14}\\
&\quad - q^{13} - 3q^{12} - 3q^{11} - 3q^{10} - 2q^9 + 2q^8\\
&\quad + 4q^7 + q^6 + 2q^5 - q^3 - 3q^2 - q + 2,\\
&q^{28} - 3q^{27} - 2q^{26} + q^{24} + 2q^{23} + q^{22} + 3q^{21}\\
&\quad + 2q^{20} - q^{19} - 3q^{18} - 4q^{17} - 4q^{16} + 2q^{13}\\
&\quad + 5q^{12} + 3q^{11} + 2q^{10} - q^9 - 3q^8 - 2q^7 - q^6\\
&\quad - 2q^5 + q^3 + 3q^2 + q - 2,\\
&q^{36} - 3q^{35} - 2q^{34} + q^{32} + 2q^{31} + q^{30} + 2q^{29}\\
&\quad + q^{28} + 2q^{27} - q^{26} - 4q^{25} - 5q^{24} - 2q^{23} - q^{22}\\
&\quad - q^{21} + 3q^{20} + 4q^{19} + 5q^{18} + 3q^{17} + q^{16} - 2q^{15}\\
&\quad - q^{14} - 4q^{13} - 5q^{12} - 2q^{11} + q^{10} + 2q^9 + q^8\\
&\quad + 2q^7 + q^6 + 2q^5 - q^3 - 3q^2 - q + 2
\end{aligned}$$

for $k = 0, \ldots, 8$. It is tedious but easy to check that all these expression evaluate to a positive number for $q \geqslant 4$.

The inductive step. We shall show

$$\prod_{i=1}^{k} \frac{q^i}{q^i - 1} \leqslant 1 + \alpha \frac{q^{k-1} - 2}{q^k - 2} \tag{2.3}$$

for $k \geqslant 8$ by induction. This is easy to verify for $k = 8$. For $k > 8$ we have

$$\begin{aligned}\prod_{i=1}^{k+1} \frac{q^i}{q^i - 1} &\leqslant \left(1 + \alpha \frac{q^{k-1} - 2}{q^k - 2}\right) \frac{q^{k+1}}{q^{k+1} - 1} \\ &= 1 + \alpha \frac{q^k - 2}{q^{k+1} - 2} - \\ &\begin{cases} \frac{47 \cdot 2^{2k} + 103 \cdot 2^{2+k} - 380}{2^5 (2^k - 2)(2^k - 1)(2^{k+1} - 1)} & \text{if } q = 2 \text{ and } \alpha = \frac{79}{16}, \\ \frac{2(3^{2k+2} + 10 \cdot 3^k - 8)}{(3^k - 2)(3^{k+1} - 2)(3^{k+1} - 1)} & \text{if } q = 3 \text{ and } \alpha = 3, \\ \frac{(q^{k+1} - 2q^k + 2)(4q^{k+1} - q^k - 6)}{(q^k - 2)(q^{k+1} - 2)(q^{k+1} - 1)} & \text{if } q \geqslant 4 \text{ and } \alpha = 2. \end{cases}\end{aligned}$$

Hence (2.3) yields the assertion, since

$$1 + \alpha \frac{q^{k-1} - 2}{q^k - 2} \leqslant 1 + \alpha q^{-1}.$$

The last assertion follows directly from the definition of the Gaussian coefficient, since

$$[n] = \frac{q^n - 1}{q - 1} < q^{n-1} \frac{q}{q - 1}.$$

$\square$

Lemma 2.2. *Let* n *be a natural number. Let* $0 < k < n$. *Then*

$$(1 + q^{-1}) q^{k(n-k)} \leqslant \begin{bmatrix} n \\ k \end{bmatrix}.$$

If $q = 2$, *then*

$$(2 - \frac{1}{q^{n-k}}) q^{k(n-k)} \leqslant \begin{bmatrix} n \\ k \end{bmatrix}.$$

Proof. We have

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix} &= \prod_{i=1}^{k} \frac{q^{n-k+i}-1}{q^i-1} \\
&\geqslant \frac{q^{n+1-k}-1}{q-1} \prod_{i=2}^{k} \frac{q^{n-k+i}-1}{q^i-1} \\
&\geqslant \frac{q^{n+1-k}-1}{q-1} \prod_{i=2}^{k} \frac{q^{n-k+i}}{q^i} \\
&\geqslant \frac{q^{n+1-k}-1}{q-1} \cdot \frac{1}{q^{n-k}} \cdot q^{k(n-k)}.
\end{aligned}$$

This shows both statements. □

Lemma 2.3. *For* $0 \leqslant k \leqslant a \leqslant n-k$ *we have*

$$q^{a(k-1)} \begin{bmatrix} n-a-1 \\ k-1 \end{bmatrix} \geqslant \left(1 - \frac{2}{q^{n-k-a+1}}\right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Proof. We may assume $k \geqslant 1$, since in the case $k = 0$ both sides of the equation are zero. We also suppose $n \geqslant 2$, since $n = 1$ implies $k = 0$. We shall show the statement by counting subspaces in $\mathbb{F}_q^n$. Let P be a point and S an a-dimensional subspace which intersects P trivially. Count the number of k-dimensional subspaces that contain P and intersect S trivially. There are $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ k-dimensional subspaces on P and at most $[a]\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$ of them meet S non-trivially. By Corollary 1.27, we have

$$q^{a(k-1)} \begin{bmatrix} n-a-1 \\ k-1 \end{bmatrix} \geqslant \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - [a] \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}. \tag{2.4}$$

Furthermore, the following inequality can be easily verified under the hypothesis and $k \geqslant 1$.

$$\begin{aligned}
&\frac{[a]\begin{bmatrix}n-2\\k-2\end{bmatrix}}{\begin{bmatrix}n-1\\k-1\end{bmatrix}} = \frac{(q^a-1)(q^{k-1}-1)}{(q^{n-1}-1)(q-1)} \\
&= \frac{2}{q^{n-a-k+1}} \\
&- \frac{q^{k+a+1}-2q^{k+a}+q^{k+1}+q^{a+2}-q^2-2q^{-n+k+a+2}+2q^{-n+k+a+1}}{q(q-1)(q^n-q)} \\
&\leqslant \frac{2}{q^{n-a-k+1}} - \frac{q^{k+1}+q^{a+2}-q^2-2q^{-n+k+a+1}(q-1)}{q(q-1)(q^n-q)} \\
&\leqslant \frac{2}{q^{n-a-k+1}} - \frac{q^2+q^3-q^2-2q(q-1)}{q(q-1)(q^n-q)} \\
&\leqslant \frac{2}{q^{n-a-k+1}}.
\end{aligned} \tag{2.5}$$

The equations (2.4) and (2.5) yield the assertion. □

2.2 BOUNDS ON GENERATORS

In this part we give upper and lower bounds on the number of generators of polar space, i.e. we provide bounds on products of the form

$$\prod_{i=a}^{b}(1+c^{-i}),$$

where $c > 1$. While the proofs work in far more general settings, we avoid generalizations that are unnecessary for this work. We start with this bound which is folklore.

Lemma 2.4 ([76])**.** *Let $x \geqslant 0$. Then we have*

$$\frac{2x}{2+x} \leqslant \log(1+x) \leqslant \frac{x}{2}\cdot\frac{2+x}{1+x}.$$

Lemma 2.5. *Let $q \geqslant 2$. The function $f : [0,\infty) \to \mathbb{R}$ defined by*

$$f(x) = \log(1+q^{-x})/\log(1+q^{-x-1}).$$

Then the first derivative f' *of* f *is bounded by*

$$f'(x) \geqslant c \cdot (q^x(2q-2)-1),$$

where

$$c = \frac{\log(q)}{\log(1+q^{-x-1})^2(q^x+1)(q^{x+1}+1)(4q^{2x+1}+2q^{x+1})}$$

Particularly, f *is monotonically increasing in* x*, i.e.* $f'(x) > 0$ *for all* $x \in (0,\infty)$.

Proof. Consider the first derivative of $\log(1+q^{-x})/\log(1+q^{-x-1})$ which is, by Lemma 2.4, $q \geqslant 2$, and $x \geqslant 0$,

$$\frac{\log(q)\left((q^x+1)\log(1+q^{-x})-(q^{x+1}+1)\log(1+q^{-x-1})\right)}{\log(1+q^{-x-1})^2(q^x+1)(q^{x+1}+1)}$$
$$\geqslant c \cdot (q^x(2q-2)-1) > 0.$$

Here we used the upper and lower bounds provided in Lemma 2.4 to bound $(q^x+1)\log(1+q^{-x})-(q^{x+1}+1)\log(1+q^{-x-1})$ by $q^x(2q-2)-1$. □

Corollary 2.6. *Let* $\alpha > 1$ *be a real number with*

$$\alpha \log(1+q^{-x-1}) \leqslant \log(1+q^{-x}).$$

Let $q \geqslant 2$*. Then the function* $g : [0,\infty) \times [2,\infty) \to \mathbb{R}$ *defined by*

$$g(x,q) = (1+q^{-x})^{\frac{\alpha}{\alpha-1}}$$

is monotonically decreasing in x *and* q.

Proof. Fir we prove the claim for x. We have that

$$\frac{\alpha-1}{\alpha} = 1 - \frac{\log(1+q^{-x-1})}{\log(1+q^{-x})}.$$

By Lemma 2.5, the first derivative of $\frac{\alpha}{\alpha-1}$ with respect to x is positive. By $q \geqslant 2$, $1+q^{-x}$ decreases if we increase x. Hence, g is monotonically decreasing in x.

By $\log(1+q^{-x}) > \log(1+q^{-x-1})$, the assertion for q follows. □

Lemma 2.7. *Let $\mathcal{P}$ be a polar space of rank d and type e.*

(a) The polar space $\mathcal{P}$ contains at least

$$q^{de+\binom{d}{2}}$$

generators.

(b) Let $\alpha > 1$ be a real number with

$$\alpha \log(1+q^{-e-1}) \leqslant \log(1+q^{-e}).$$

Let x be the number of generators of $\mathcal{P}$. Then

$$x \cdot q^{-de-\binom{d}{2}} = \prod_{i=0}^{d-1}(1+q^{-e-i}) \leqslant \left(1+q^{-e}\right)^{\frac{\alpha}{\alpha-1}}.$$

Furthermore, the second inequality holds for all $e \in \mathbb{R}$.

Proof. The first claim is trivial consequence of (1.17). We shall prove the second claim in the following.

By Lemma 2.5, the hypothesis on α implies

$$\alpha \log(1+q^{-e-1-i}) \leqslant \log(1+q^{-e-i})$$

for all $i \geqslant 0$. Then

$$\sum_{i=0}^{\infty} \log(1+q^{-e-i}) \leqslant \sum_{i=0}^{\infty} \alpha^{-i} \log(1+q^{-e})$$
$$= \log(1+q^{-e}) \sum_{i=0}^{\infty} \alpha^{-i} = \log(1+q^{-e}) \frac{\alpha}{\alpha-1}.$$

Hence,

$$\prod_{i=0}^{d-1}(1+q^{-e-i}) \leqslant \left(1+q^{-e}\right)^{\frac{\alpha}{\alpha-1}}.$$

$\square$

Particularly, the upper bound in the previous result is noteworthy as it is much tighter for many choices of q and e than the standard upper bound of

$$\prod_{i=0}^{d-1}(1+q^{-e-i}) < 2+\frac{1}{q^e}.$$

See [66, Lemma 11] for a proof of this standard bound. Alternatively, one can prove bounds like these by induction. The following is an example for this.

Lemma 2.8. *Let* $q \geqslant 2$*. Let* $d \geqslant 1$*. Then*

$$\prod_{i=1}^{d-1}(q^i+1) \leqslant \frac{2q^d}{q^d+1}\left(q^{\binom{d}{2}} - q^{\binom{d-1}{2}} + 1\right) + q^{\binom{d-2}{2}+2(d-2)}.$$

Proof. We prove the assertion by induction on d. It can be easily checked that assertion is true for $d \leqslant 4$. If the assertion is true for $d \geqslant 4$, then

$$\begin{aligned}\prod_{i=1}^{d}(q^i+1) &\leqslant (q^d+1)\left(\tfrac{2q^d}{q^d+1}\left(q^{\binom{d}{2}}-q^{\binom{d-1}{2}}+1\right)+q^{\binom{d-2}{2}+2(d-2)}\right)\\ &\overset{(*)}{\leqslant} \frac{2q^{d+1}}{q^{d+1}+1}\left(q^{\binom{d+1}{2}} - q^{\binom{d}{2}} + 1\right) + q^{\binom{d-1}{2}+2(d-1)}.\end{aligned}$$

The difference between the right hand side of (*) and the left hand side of (*) equals

$$\begin{aligned}\frac{q^d}{q^{d+1}+1}(2q^{\binom{d}{2}+2} &- 2q^{\binom{d}{2}+1} - 3q^{\binom{d}{2}}\\ &+ 2q^{\binom{d-1}{2}} - q^{\binom{d-1}{2}-2} - 2q^{d+1} + 2q - 2),\end{aligned}$$

which is a positive expression for $q \geqslant 2$ and $d \geqslant 4$. $\square$

3 EKR SETS

The aim of this chapter is to give an overview over the state of the art on various topics related to Erdős-Ko-Rado sets (*EKR sets*) without the new results presented in this thesis.

3.1 EKR SETS

In the most restrictive definition, an EKR set is a set of k-subsets of $\{1, \ldots, n\}$ which pairwise intersect non-trivially. Alternatively, an EKR set is a $\{1, \ldots, d-1\}$-clique of $J(n, k)$. Erdős, Ko, and Rado proved the following famous result in [32].

Theorem 3.1 (Theorem of Erdős, Ko, and Rado)**.** *Let be* $n \geqslant 2k$*. Let* Y *be an EKR set of* k*-element subsets of* $\{1, \ldots, n\}$*. Then*

$$|Y| \leqslant \binom{n-1}{k-1}$$

with equality for $n > 2k$ *if and only if* Y *is set of all* k*-sets containing a fixed element.*

There exist many generalizations of this problem. One type of generalization changes the conditions on Y, another type of generalizations changes the association scheme from $J(n, k)$ to something else. Relevant for this thesis are analogs of the following problems on $J(n, k)$ in vector spaces and polar spaces. We state all these results for distance-regular graphs with diameter d.

(a) The largest t-intersecting families. Here Y is a $\{1, \ldots, t\}$-clique.

(b) The largest cross-intersecting families. Here one has sets Y and Z such that each element in Y has at most distance $d-1$ from each element in Z. The meaning of the word *largest* is that we want to maximize $|Y| \cdot |Z|$.

(c) The dual problem, a $\{d\}$-clique.

(d) A mysteriously related problem, the so-called Manickam-Miklós-Singhi conjecture (MMS conjecture).

We will discuss all of these different generalization for the Johnson scheme, the Grassmann scheme, and the schemes of the generators of dual polar graphs. Nearly all of the stated results can be proven with Proposition 1.9 and Proposition 1.12.

3.1.1 *The Largest* t*-Intersecting EKR Sets*

Already Erdős, Ko, and Rado considered the natural generalization of the problem to $\{1, \ldots, t\}$-cliques in [32], but there were unable to give tight bounds for these cases. Tight bounds were first given by Frankl [34], and by Wilson [80].

Theorem 3.2. *Let be* $n \geqslant 2k$*. Let* Y *be a* t*-intersecting EKR set of* k*-element subsets of* $\{1, \ldots, n\}$*. Then*

$$|Y| \leqslant \max\left\{ \binom{n-t}{k-t}, \binom{2k-t}{k} \right\}.$$

The case of equality was first classified by Wilson [80] when $t = d-1$, and by Ahlswede and Khachatrian [2] for general t.

3.1.2 *Cross-Intersecting EKR Sets*

There exists the following modification of the original EKR problem which attracted a lot of interest: a *cross-intersecting EKR set* is a pair (Y, Z) of sets of subsets with k elements of $\{1, \ldots, n\}$ such that all $y \in Y$ and $z \in Z$ intersect non-trivially. If one wants to generalize the theorem of Erdős, Ko, and Rado to this structure, then the following question arises: how do we measure the size of (Y, Z)? There are at least two natural choices. Either one goes for an upper bound for $|Y| + |Z|$ or one tries to find the upper bound for $|Y| \cdot |Z|$. In the set case the first project was pursued in [42], while the second one was completed in [62]. It turns out in the set case that the largest example satisfies $Y = Z$ and Y is one of the largest EKR sets.

3.1.3 *Partitions*

The dual problem, a $\{d\}$-clique, is a partial partition of the set $\{1, \ldots, n\}$ into k-element sets. Obviously, the largest examples have size $\lfloor \frac{n}{k} \rfloor$.

3.1.4 *The MMS Conjecture*

The MMS conjecture is motivated by the *first distribution invariant* of an association scheme $(X, \mathcal{R})$. The concept of the i*-th distribution invariant* was introduced by Bier and Delsarte [8, 9]. We say that a vector v is *general* if and only if $v^T \chi_x \neq 0$ for all $x \in X$. The i*-th distribution invariant* of an association scheme is defined as

$$\min_{w \in V_i \text{ general}} |\{x : w^T \chi_x > 0\}|.$$

In other words, we put weights on the vertices of the association scheme such that we minimize the number of vertices with postive weights. Hereby we have the restriction that the weight vector has to lie in the i-th eigenspace and is general. The first distribution invariant of $J(n, k)$ was calculated by Bier and Manickam [10].

It turns out that the determination of the first distribution invariant for the Johnson scheme is nearly identical to the following problem as was noticed by Manickam, Miklós, and Singhi [60, 61]. Let $f : X \to \mathbb{R}$ be a weight function with $\sum_{x \in X} f(x) = 0$. What is the minimum number of k-element subsets S such that $\sum_{x \in S} f(x)$ is nonnegative? Manickam, Miklós, and Singhi conjectured the following.

Conjecture 3.3 ([61, Conjecture 1.4])**.** *Let $n \geqslant 4k$. The number of k-element subsets S such that $\sum_{x \in S} f(x)$ is nonnegative is at least $\binom{n-1}{k-1}$. In the case of equality, the nonnegative subsets are all k-element subsets on a fixed element.*

The connection to Theorem 3.1 is obvious, as the bound is the same and the examples reaching this bound are the same. At the time of writing the MMS conjecture is a quite popular topic. We shall mention some important works on this conjecture such as a result by Alon, Huang, and Sudakov [3] who obtained the first polynomial bound on n with $n \geqslant \min\{33k^2, 2k^3\}$, a result by Pokrovskiy [67] who obtained the first linear bound on k with $n > 10^{46}k$, and a result by Chowdhury, Sarkis, and Shahriari [16] who did prove the conjecture for $n \geqslant 8k^2$ and also obtained a result on the analog problem on vector spaces.

3.2 EKR SETS OF VECTOR SPACES

Define the following examples for t-intersecting EKR sets. The names are only used in this thesis, but they are inspired by the work of Irit Dinur and Ehud Friedgut [27]. Their main purpose is that they make the similarities between large EKR sets of vector spaces and polar spaces more obvious.

(a) We call the set of all k-dimensional subspaces, which meet a fixed s-space at least in dimension $(s+t)/2$, an (s,t)-*junta*.

(b) We call a (t,t)-junta a t-*dictatorship*.

We call a 1-dictatorship *dictatorship* and a $(s,1)$-junta s-*junta*.

Other words used for a dictatorship include sunflower, star, and point-example. The EKR theorem for vector spaces is as follows.

Theorem 3.4 ([46]). *Let be $n \geqslant 2k$. Let Y be an EKR set of k-dimensional subspaces of $\mathbb{F}_q^n$. Then*

$$|Y| \leqslant \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

If the bound is tight, then one of the following occurs:

(a) Y *is a dictatorship,*

(b) $n = 2k$ *and* Y *is a* $(2k-1)$*-junta.*

The most part of the classification was done by Hsieh [46] with a few exceptions. Frankl and Wilson [35] classified all largest examples when $n > 2k$. The largest examples for $n = 2k$ became folklore after 1986, one proof can be found in Newman's PhD thesis [63].

3.2.1 *The Largest* t*-Intersecting EKR Sets*

The t-intersecting EKR theorem for vector spaces was first given by Frankl and Wilson who also classified nearly all examples in their famous paper [35]. All largest examples were classified in a very elegant proof by Tanaka [72].

Theorem 3.5 ([35, 72])**.** *Let be* $n \geqslant 2k-1+t$*. Let* Y *be an EKR set of* k*-dimensional subspaces of* $\mathbb{F}_q^n$*. Then*

$$|Y| \leqslant \max\left\{ \begin{bmatrix} n-t \\ k-t \end{bmatrix}, \begin{bmatrix} 2k-t \\ k \end{bmatrix} \right\}.$$

If the bound is tight, then one of the following cases occurs:

(a) Y *is a* t*-dictatorship,*

(b) $n = 2k-1+t$ *and* Y *is a* $(2k-t, t)$*-junta.*

3.2.2 *Cross-Intersecting EKR Sets*

Cross-t-intersecting EKR sets (i.e. cross-intersecting EKR sets (Y, Z) with pairwise intersections in at least dimension t) were studied recently by Tokushige [75]. The largest examples satisfy $Y = Z$ and are exactly the largest t-intersecting EKR sets. Related results are due to Suda and Tanaka [71] who did classify the largest examples of cross-intersecting EKR sets (Y, Z) with the difference that Y is a set of k-spaces and Z is a set of s-spaces.

3.2.3 *Spreads*

The dual problem, finding large $\{d\}$-cliques of $J_q(n, q)$, is well-known in finite geometry. These cliques are called *(partial) spreads*.

Theorem 3.6 ([7, Result 2.1])**.** *The vector space* $\mathbb{F}_q^n$ *contains a spread of* k*-dimensional subspaces if and only if* k *divides* n*.*

Note that there are things similar to a spread if k does not divide n.

Theorem 3.7 (Beutelspacher [7, Theorem 4.2]). *The vector space $V = \mathbb{F}_q^n$ with $n = bk + r$ and $0 \leqslant r \leqslant k - 1$. Let S be a $(k + r)$-dimensional subspace of V. Then there exists a partition of $V \setminus S$ into k-dimensional subspaces.*

Notice that one can easily add one k-dimensional subspaces $T \subseteq S$ to the set of k-dimensional subspaces described in Theorem 3.7. In particular, [7, Theorem 4.1] shows that the example in Theorem 3.7 is a $\{d\}$-clique of maximum size.

3.2.4 *The MMS Conjecture*

Manickam and Singhi considered the analog problem to the MMS conjecture on sets for vector spaces [61]. In this case the problem can be paraphrased as follows. Let V be a finite n-dimensional vector space. Let $\mathcal{P}$ be the set of 1-dimensional subspaces of V. Let $f : \mathcal{P} \to \mathbb{R}$ be a weight function with $\sum_{P \in \mathcal{P}} f(P) = 0$. What is the minimum number of k-dimensional subspaces S such that $\sum_{P \in S} f(P)$ is nonnegative? They conjectured the following.

Conjecture 3.8 ([61, Conjecture 1.4]). *Let $n \geqslant 4k$. The number of k-dimensional subspaces S such that $\sum_{P \in \mathcal{P}} f(P)$ is nonnegative is at least the size of a dictatorship, i.e. the number of k-dimensional subspaces on a fixed 1-dimensional subspace.*

Manickam and Singhi were able to prove their conjecture if k divides n (which includes $n = k, 2k, 3k$) [61]. Recently, Chowdhury, Huang, Sarkis, Shahriari, and Sudakov showed that Conjecture 3.8 holds for $n \geqslant 3k$ [47, 16]. Hence, technically Conjecture 3.8 is proven, but all known counterexamples satisfy $k < n < 2k$, so it seems reasonable to conjecture that only $n \geqslant 2k$ is necessary. We shall extend the technique used by Chowdhury, Sarkis, and Shahriari to show the conjecture for $n \geqslant 2k$ and (very) large q (Theorem 8.1) as well as reduce the analog problem for $k \leqslant n < 2k$ to the case $2k \leqslant n$ (Lemma 8.14).

Note that also in this case there is an analog to the first distribution invariant of $J_q(n, k)$. See [61] for details.

3.3 EKR SETS OF POLAR SPACES

In the case of polar spaces of rank d, we call $(d-t)$-intersecting EKR sets (d,t)-EKR sets. Define the following examples for EKR sets.

(a) We call the set of all generators, which meet a fixed totally isotropic s-space at least in dimension $(s+d-t)/2$, an (s,t)-*junta*.

(b) We call a $(t, d-t)$-junta a t-*dictatorship*.

If σ is an isomorphism of a polar space and Y is a largest EKR set of the same polar space, then Y^σ is a largest EKR set. Hence the following results are classifications up to isomorphism even if we do not mention this explicitly.

Theorem 3.9 (Stanton [68]). *A set of pairwise non-trivially intersecting generators of a finite classical polar space has at most the size of a dictatorship with the following exceptions:*

(a) In the case $Q^+(2d-1,q)$, d *odd, the set of all Latin generators is a largest example.*

(b) The Hermitian polar space $H(2d-1,q^2)$, d *odd.*

Theorem 3.10 (Pepe, Storme, and Vanhove [66]). *A largest set of pairwise non-trivially intersecting generators of a finite classical polar space is a dictatorship with the following exceptions:*

(a) In the case $Q^+(2d-1,q)$, d *odd, the set of all Latin (or Greek) generators is the unique largest example.*

(b) In the case $Q(2d,q)$, d *odd, the set of all Latin (or Greek) generators of a* $Q^+(2d-1,q)$ *is another largest example. If* $d=3$*, then a 3-junta is another largest example. These are all examples.*

(c) The case $W(2d-1,q)$, q *even, is as the case* $Q(2d,q)$.

(d) In the case $H(5,q^2)$ *a 3-junta is the only largest example.*

(e) The Hermitian polar space $H(2d-1,q^2)$, d *odd,* $d>3$.

In the last case of the previous theorem, the largest examples are unknown. The results in this thesis are basically the first on t-intersecting EKR sets and cross-intersecting EKR sets, so we can not state any previous results.[1]

3.3.1 *(Partial) Spreads*

A partial spread is called a *spread* of generators of a polar space if it is a partition of the points of the polar space. In general, not that much is known about spreads of polar spaces. We just mention some existence results in Table 1.

We prove a new upper bound on partial spreads in $H(2d-1, q^2)$, $d \geqslant 4$ even, in Chapter 7.

3.3.2 *The MMS Conjecture*

In this case the problem can be paraphrased as follows. Let $\mathcal{P}$ be the set of points of a polar space. Let $f : \mathcal{P} \to \mathbb{R}$ be a weight function with $\sum_{P \in \mathcal{P}} f(P) = 0$. What is the minimum number of generators S such that $\sum_{P \in S} f(P)$ is nonnegative? Nothing is known about the MMS conjecture for polar spaces, not even a conjecture. The author assumes that again dictatorships give lower bounds on the minimum number of nonnegative subspaces. The only reason to mention the conjecture is that one can easily apply the proof by Manickam and Singhi [61] for vector spaces which did show the following.

Theorem 3.11 ([61, Theorem 3.1])**.** *Suppose that* k *divides* n*. Then the number of* k*-dimensional subspaces such that* $\sum_{P \in \mathcal{P}} f(P)$ *is nonnegative is at least the number of* k*-dimensional subspaces on a fixed* 1*-dimensional subspace. If equality holds, then the nonnegative* k*-dimensional subspaces are an EKR set.*

The same proof can be used to derive the following which is unpublished, but not very interesting for everybody who knows the proof of [61, Theorem 3.1].

1 See the corresponding chapters for restrictions on this claim.

Polar Space	Existence of Spreads
$W(2d-1,q)$	yes
$Q(6,q)$	$q=3^h$: yes $q=2^h$: yes $q \geqslant 5$ prime: yes $q = p^h$, $p \equiv 2 \mod 3$, p prime, h odd: yes
$Q(2d,q)$	$d=2$: yes q even: yes $d>2$ even, q odd: no
$Q^-(2d+1,q)$	$d=2$: yes q even: yes
$Q^+(2d-1,q)$	$d=2$: yes $d=4$: yes if and only if $Q(6,q)$ has a spread d even, q even: yes d odd: no
$H(2d-1,q^2)$	no
$H(4,4)$	no

Table 1: The existence of spreads according to [19].

Theorem 3.12. *Suppose that a polar space $\mathcal{P}$ has a spread. Then the number of generators such that $\sum_{P\in\mathcal{P}} f(P)$ is nonnegative is at least the size of the largest EKR set of the polar space.*

Hence, for all cases where there is a yes in Table 1, the problem is simple. Otherwise the problem seems to be complicated.

3.4 CONCLUDING REMARKS

There are many more interesting results on EKR sets in various settings. This list is far from complete. The author wants to point out some more nice geometrical results, for example on EKR sets in vector spaces and projective spaces on planes [21] and Hilton-Milner type Theorems for $Q^+(4n+1, q)$ [23]. There also other structures where EKR problems are heavily investigated. One very popular example are permutation groups [15, 57, 37, 31]. Other examples include particular designs [22].

4

EKR SETS OF HERMITIAN POLAR SPACES

Prior to the result presented in this chapter, the best known upper bound on EKR sets Y in $H(2d-1, q^2)$, $d > 3$ odd, was approximately $|Y| \leqslant q^{d(d-1)}$ [68]. In this chapter set

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_{q^2}.$$

We shall prove the following.

Theorem 4.1. *Let Y be an EKR set of generators on $H(2d-1, q^2)$, $d \geqslant 3$ odd. Then*

$$|Y| \leqslant \frac{nq^{d-1} - f_1(q^{d-1}-1)(1-c)}{q^{2d-1} + q^{d-1} + f_1(q^{d-1}-1)c} \approx q^{(d-1)^2+1}$$

where the constants n, f_1, c are defined by $n = \prod_{i=0}^{d-1}(q^{2i+1}+1)$, $f_1 = q^2[d]\frac{q^{2d-3}+1}{q+1}$ and $c = \frac{q^2-q-1+q^{-2d+3}}{q^{2d}-1}$.

4.1 PROOF OF THEOREM 4.1

We start by explicitly stating some parameters of the dual polar graph of $H(2d-1, q^2)$. By (1.17), we have

$$n = \prod_{i=0}^{d-1}(q^{2i+1}+1).$$

By Theorem 1.23, we have

$$n_s = P_{0,s} = q^{s^2}\begin{bmatrix} d \\ s \end{bmatrix},$$
$$P_{1,s} = \begin{bmatrix} d-1 \\ s \end{bmatrix} q^{s^2} - \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{(s-1)^2},$$
$$P_{r,d} = (-1)^r q^{2(r-1)^2+(3-2d)(r-1)+(d-1)^2}.$$

By Theorem 1.22, we have

$$f_1 = q^2[d]\frac{q^{2d-3}+1}{q+1}.$$

By Lemma 1.6 and the previous calculations, we have

$$\begin{aligned}
Q_{s,1} &= n_s^{-1} P_{1,s} f_1 \\
&= \frac{f_1 \left(\left[{d-1 \atop s}\right] q^{s^2} - \left[{d-1 \atop s-1}\right] q^{(s-1)^2} \right)}{q^{s^2} \left[{d \atop s}\right]} \\
&= f_1 \frac{q^{2(d-s)} - q - 1 + q^{-2s+1}}{q^{2d} - 1}.
\end{aligned}$$

In particular,

$$\begin{aligned}
Q_{0,1} &= f_1, \\
Q_{d-1,1} &= f_1 \frac{q^2 - q - 1 + q^{-2d+3}}{q^{2d} - 1},
\end{aligned}$$

and for $d - 1 > s > 0$

$$\begin{aligned}
&(q^{2d} - 1) \frac{Q_{s,1} - Q_{d-1,1}}{f_1} \\
&= q^{2(d-s)} - q^{-2d+3} + q^{-2s+1} - q^2 \qquad (4.1) \\
&\geqslant q^4 - q^{-1} - q^2 > 0.
\end{aligned}$$

Let Y be a subset of X and χ the characteristic vector of Y. Put $y := |Y|$. Define the inner distribution $a = (a_0, a_1, \ldots, a_d)$ of Y as in Subsection 1.4.2. We have $a_0 = 1$ and $\sum_{i=0}^{d} a_i = y$. Furthermore, Lemma 1.13 and Proposition 1.14 show that

$$(yaQ)_j = n\chi^T E_j \chi \geqslant 0. \qquad (4.2)$$

Recall that

$$\sum_{j=0}^{d} E_j = I, \qquad E_0 = n^{-1} J, \qquad A_j = \sum_{i=0}^{d} P_{ij} E_i$$

Note that $P_{d,d}$ is the minimum of $\{P_{i,d} : i = 0, \dots, d\}$ if d odd. By these observations and $d \geqslant 3$ odd, we have

$$\begin{aligned}0 &\leqslant n\chi^T \left(\sum_{i=2}^{d} (P_{i,d} - P_{d,d})E_i\right)\chi \\ &= n\chi^T\left(A_d - P_{0,d}E_0 - P_{1,d}E_1\right)\chi - nP_{d,d}\chi^T\left(I - E_0 - E_1\right)\chi \\ &= n\chi^T A_d\chi - nP_{d,d}\chi^T\chi - (P_{0,d} - P_{d,d})\chi^T J\chi \\ &\qquad\qquad - n(P_{1,d} - P_{d,d})\chi^T E_1\chi \\ &= n\chi^T A_d\chi - nP_{d,d}y - (P_{0,d} - P_{d,d})y^2 - y(P_{1,d} - P_{d,d})(aQ)_1.\end{aligned}$$

Let Y be an EKR set. By definition $\chi^T A_d\chi = 0$. Hence the previous inequality can be rewritten as

$$(P_{0,d} - P_{d,d})y + (P_{1,d} - P_{d,d})(aQ)_1 \leqslant -nP_{d,d}.$$

As Y is an EKR set, we have $a_d = 0$. Hence,

$$\begin{aligned}&a_0 = 1, \text{ and} \\ &a_{d-1} = y - 1 - \sum_{i=1}^{d-2} a_i.\end{aligned}$$

By (4.1), $Q_{i,1} - Q_{d-1,1} > 0$ for $d-1 > i > 0$. Hence,

$$\begin{aligned}(aQ)_1 &= \sum_{i=0}^{d-1} Q_{i,1}a_i \\ &= Q_{0,1} + Q_{d-1,1}a_{d-1} + \sum_{i=1}^{d-2} Q_{i,1}a_i \\ &= Q_{0,1} + Q_{d-1,1}\left(y - 1 - \sum_{i=1}^{d-2} a_i\right) + \sum_{i=1}^{d-2} Q_{i,1}a_i \\ &= Q_{0,1} - Q_{d-1,1} + Q_{d-1,1}y + \sum_{i=1}^{d-2}(Q_{i,1} - Q_{d-1,1})a_i \\ &\geqslant Q_{0,1} - Q_{d-1,1} + Q_{d-1,1}y.\end{aligned}$$

Thus we obtain the inequality

$$\begin{aligned}(P_{0,d} - P_{d,d})y + (P_{1,d} - P_{d,d})(Q_{d-1,1}y + (Q_{0,1} - Q_{d-1,1})) \\ \leqslant -nP_{d,d}.\end{aligned}$$

This can be rewritten as

$$
\begin{aligned}
y &\leqslant \frac{-nP_{d,d} - (P_{1,d} - P_{d,d})(Q_{0,1} - Q_{d-1,1})}{P_{0,d} - P_{d,d} + (P_{1,d} - P_{d,d})Q_{d-1,1}} \\
&= \frac{nq^{d-1} - f_1(q^{d-1} - 1)\left(1 - \frac{q^2-q-1+q^{-2d+3}}{q^{2d}-1}\right)}{q^{2d-1} + q^{d-1} + f_1(q^{d-1} - 1)\frac{q^2-q-1+q^{-2d+3}}{q^{2d}-1}}.
\end{aligned}
$$

As $n \approx q^{d^2}$ and $f_1 \approx q^{4(d-1)}$, the largest term in the nominator is $nq^{d-1} \approx q^{d^2+d-1}$, and the largest term in the denominator is $q^{3(d-1)}$. Hence, the bound on y is approximately $q^{(d-1)^2+1}$, as stated in the theorem.

4.2 CONCLUDING REMARKS

The general idea is to take the inner distribution vector a and the $d+1$ linear inequalities

$$(aQ)_j \geqslant 0$$

and solve the corresponding linear optimization problem. The matrix Q is implicitly given by Theorem 1.23 and Lemma 1.6, so it is easy to calculate an optimal solution for arbitrary d. For $d = 3$ this yields

$$y \leqslant q^5 + q^4 + q^3 + 1.$$

The new result shows the same inequality. In [66] it is proven that the sharp upper bound is

$$y \leqslant q^5 + q^3 + q + 1.$$

For $d = 5$ linear optimization yields

$$y \leqslant q^{17} + 2q^{16} + 2q^{15} + q^{14} + q^{10} + 2q^9 + 3q^8 + 2q^7 + q + 1.$$

The new result shows

$$
\begin{aligned}
y \leqslant\ & q^{17} + 3q^{16} + 4q^{15} + 5q^{14} + 7q^{13} + 9q^{12} \\
& + 11q^{11} + 12q^{10} + 12q^9 + 9q^8 + 3q^7 - 7q^6 \\
& - 19q^5 - 35q^4 - 55q^3 - 77q^2 - 97q - 111.
\end{aligned}
$$

The largest known examples for EKR sets of generators in $H(9,q^2)$ is a dictatorship, i.e. the set of all generators through a point. The second largest known example is a d-junta, i.e. the set of generators that meet a given generator in at least a plane, this selected generator included. Both examples have a size of approximately q^{16}. Hence it seems reasonable to assume that this algebraic approach is not able to give the correct upper bound. Together with Klaus Metsch the author conjectured that a dictatorship is the largest EKR set of generators on $H(2d-1,q^2)$ for $d > 3$.

As a conclusion it does not seem worth to prove the best possible result with linear programming for general d for the following reasons.

(a) The given proof is much easier than a proof of the linear programming bound.

(b) The given bound is nearly as good as the expected bound through linear programming.

(c) The best possible bound through linear programming is most likely wrong about the factor q for $d > 3$.

We give an alternative proof of Theorem 4.1 in Chapter 6. This alternative proof relies on the same idea and is a tad more general, but the author did not check if it yields exactly the same bounds. The bound in Chapter 6 is also approximately $q^{(d-1)^2+1}$.

5

(d,t)-EKR SETS OF POLAR SPACES

With two exceptions no attempts were made to investigate (d,t)-EKR sets of finite classical polar spaces until now. One exception is the investigation of $(d,d-1)$-EKR sets which is the topic of other parts of this thesis. The other exception is the investigation of $\{0,1,2\}$-cliques of dual polar graphs by Brouwer and Hemmeter [14], where they classified all $\{0,1,2\}$-cliques of generators on polar spaces in the non-Hermitian case. This problem was modified by De Boeck [21] to EKR sets, where he classified EKR sets Y of planes (not necessarily generators) for $d \in \{3,4,5\}$ and $|Y| \geqslant 3q^4+3q^3+2q^2+q+1$.

Brouwer and Hemmeter investigated $(d,2)$-EKR sets of classical finite polar spaces. This paper is concerned about generalizing their work to (d,t)-EKR sets of maximum size for more values of t. We classify the largest (d,t)-EKR sets for $t \leqslant \sqrt{8d/5}-2$ if $q \geqslant 3$ and for $t \leqslant \sqrt{8d/5}-2$ if $q \geqslant 2$ (Theorem 5.12, Theorem 5.24, and Theorem 5.1). These results imply upper bounds on the size of the second largest example, so it might provide a reasonable basis to classify the second largest maximal (d,t)-EKR sets as it was done for EKR sets of sets [43], vector spaces [12], and some special cases in polar spaces [23, 21]. Furthermore, we give non-trivial upper bounds for general t, $q \geqslant 3$ (Theorem 5.33). As a side effect we determine the smallest, largest, and second largest eigenvalues of the adjacency matrix of the considered associated graph for $q \geqslant 3$ in Theorem 5.25. These numbers alone are important parameters of a graph as they can be easily used to make non-trivial statements on many other properties of the graph such as the chromatic number or the convergence of random walks.

The main result of this section is the following.

Theorem 5.1. *Let $0 < t \leqslant d$. All finite classical polar spaces of rank d satisfy the following.*

(a) **Case t even.** *The largest (d,t)-EKR set is a d-junta if one of the following conditions is satisfied:*

(i) $t \leqslant \sqrt{\frac{8d}{9}}-2$, $q \geqslant 2$,

(ii) $t \leqslant \sqrt{\frac{8d}{5}}-2$, $q \geqslant 3$.

(b) **Case t odd.** *The largest (d, t)-EKR set is a $(d-1)$-junta if one of the following conditions is satisfied:*

(i) $t \leqslant \sqrt{\frac{8d}{9}} - 2$, $q \geqslant 2$,

(ii) $t \leqslant \sqrt{\frac{8d}{5}} - 2$, $q \geqslant 3$.

Beyond these results the author hopes that the used technique, which combines algebraic and geometrical arguments, is applicable to EKR problems in other interesting structures, and can be modified to classify all (d, t)-EKR sets of generators of maximum size for more d and t.

This chapter is organized as follows. The main parts are Section 5.3 and Section 5.4, where we develop stability results for (d, t)-EKR sets in Theorem 5.12 and Theorem 5.24 which depend on the maximum size of $(2t-1, t)$- and $(2t-2, t)$-EKR sets. We use the inequalities on the Gaussian coefficient and the number of generators from Chapter 2 to approximate Hoffman's bound for (d, t)-EKR sets in Section 5.6 for $q \geqslant 3$. Finally, in Section 5.7 we prove Theorem 5.1.

5.1 SOME CONSTANTS

We define the following constants. Recall the definition of ψ_2 and ψ_3 from Section 1.6. Even if the constants depend on the type e of the polar space, we usually leave e out of the name of the polar space as we do not have interactions between polar spaces of different type (so e is always fixed).

Definition 5.2. *Let d, t, r be integers. Let $d \geqslant 0$.*

- *Let ψ^0 be for t even defined as*

$$\sum_{i=1}^{t/2-1} \psi_3(d, d - \frac{3}{2}t, d - 2t + 1, d - \frac{t}{2}, d - \frac{5}{2}t + 1 + i, d - 2t).$$

- *Let* ψ^1 *be for* t *odd defined as*

$$\sum_{i=1}^{t/2-3/2} \psi_3(d-1, d-\frac{3}{2}t+\frac{1}{2}, d-2t+2, d-\frac{1}{2}t-\frac{1}{2}, d-\frac{5}{2}t+\frac{5}{2}+i, d-2t+1)$$

- *Let* $\overline{\psi}^1$ *be* $\psi_2(d, d-\frac{t}{2}+\frac{1}{2}, d-\frac{3}{2}t+\frac{1}{2}, d-2t+1)$ *for* t *odd.*
- *Let* $\omega(d,r)$ *be the number of generators that contain a fixed* $(d-r)$*-space in a polar space of rank* d *if* $0 \leqslant r \leqslant d$*,* 0 *otherwise.*
- *Let* $c_{d,t}$ *be the maximum size of a* (d,t)*-EKR set of generators of a finite classical polar space of rank* d*.*

Lemma 5.3. *We have the following.*

(a)

$$\psi^0 = q^{\frac{3}{4}t^2} \sum_{i=1}^{t/2-1} q^{(\frac{t}{2}-1-i)(\frac{t}{2}-i)} \begin{bmatrix} d-2t+1 \\ t/2-i \end{bmatrix} \begin{bmatrix} t/2-1 \\ i \end{bmatrix},$$

(b)

$$\psi^1 = q^{(\frac{t}{2}-\frac{1}{2})(\frac{3}{2}t-\frac{3}{2})} \cdot \sum_{i=1}^{t/2-3/2} q^{(\frac{t}{2}-\frac{3}{2}-i)(\frac{t}{2}-\frac{1}{2}-i)} \begin{bmatrix} d-2t+2 \\ (t-1)/2-i \end{bmatrix} \begin{bmatrix} \frac{t}{2}-\frac{3}{2} \\ i \end{bmatrix},$$

(c)

$$\overline{\psi}^1 = q^{(\frac{3}{2}t-\frac{1}{2})(\frac{t}{2}-\frac{1}{2})} \begin{bmatrix} d-\frac{3}{2}t+\frac{1}{2} \\ (t-1)/2 \end{bmatrix},$$

(d)

$$\omega(d,r) = \prod_{i=0}^{r-1}(q^{i+e}+1).$$

Proof. Theorem 1.26 shows the first three claims.

The number $\omega(d,r)$ equals the number of generators in the quotient geometry of a $(d-r)$-space. That is a polar space of the same type with generators of rank r. The claim follows from (1.17). □

5.2 A PROPERTY OF (d,t)-EKR SETS

An (d,t)-EKR set is *maximal* if it is not a proper subset of another (d,t)-EKR set. We need the following basic result on maximal (d,t)-EKR sets.

Lemma 5.4. *Let* Y *be a* (d,t)*-EKR set where* $d \geqslant t \geqslant 0$*. If* Y *is a* $(d,t-1)$*-EKR set, then* Y *is not a maximal* (d,t)*-EKR set.*

Proof. If $t=0$, then the unique maximal $(d,-1)$-EKR set is the empty set, but every maximal $(d,0)$-EKR set is a set with one generator as its only element.

Let $t>0$. Suppose that Y is a $(d,t-1)$-EKR set.

Case 1. If there are no $a_1,a_2 \in Y$ such that $\dim(a_1 \cap a_2) = d-t+1$, then (by induction on t) Y is not a maximal $(d,t-1)$-EKR set, hence also not a maximal (d,t)-EKR set.

Case 2. There exist $a_1,a_2 \in Y$ such that $\dim(a_1 \cap a_2) = d-t+1$. Take a $(d-1)$-dimensional subspace a_1' of a_1 such that $\dim(a_1' \cap a_2) = d-t$. Then all $b \in Y$ satisfy $\dim(a_1' \cap b) \geqslant \dim(a_1 \cap b) - 1 = d-t$. Let Y' be the set of all generators through a_1'. By construction $Y \cup Y'$ is still a (d,t)-EKR set, but there exists a generator c through a_1' such that $\dim(c \cap a_2) = d-t$. Hence, Y is not maximal. □

Notice that not only, as seen in the previous result, $(d,-1)$-EKR sets and $(d,0)$-EKR sets are trivial, but also $(d,1)$-EKR sets are trivial by the following result. Particularly, the case $d=2$ is trivial.

Lemma 5.5. *The largest* $(d,1)$*-EKR set* Y *is a dictatorship.*

Proof. Suppose $|Y|>1$. Let $a,b \in Y$ such that $a \cap b$ is a subspace of dimension $d-1$. Suppose for a contradiction that there exists a $c \in Y$ with $a \cap b \cap c$ is a subspaces of dimension smaller than $d-1$. As we have $\dim(a \cap b) = \dim(a \cap c) = \dim(b \cap c) = d-1$ and $\langle a \cap b, a \cap c\rangle \cap (b \cap c) = a \cap b \cap c$, this shows

$$\begin{aligned}
&\dim(\langle a \cap b, a \cap c, b \cap c\rangle) \\
&= \dim(a \cap b) + \dim(a \cap c) + \dim(b \cap c) - 2\dim(a \cap b \cap c) \\
&= 3(d-1) - 2(d-2) = d+1.
\end{aligned}$$

As $\langle a \cap b, a \cap c, b \cap c\rangle$ is a totally isotropic subspace, this is a contradiction. Hence, if $|Y| > 1$, then all elements of Y contain a fixed $(d-1)$-dimensional subspace S. $\square$

5.3 EKR SETS, t EVEN

Throughout this section we work in a finite classical polar space of rank $d > 2$ and given type e. Also Y is a maximal (d,t)-EKR set and t is even.

Definition 5.6. *We define the constants* b_1^0 *and* b_2^0 *by*

$$b_1^0 = \begin{bmatrix} d-\frac{3}{2}t \\ t/2-1 \end{bmatrix} c_{2t-1,t}$$
$$b_2^0 = q^{e\frac{t}{2}+\binom{t/2}{2}}\psi^0.$$

Lemma 5.7. *Let* Y *be a* (d,t)*-EKR set.*

(a) Let P *be a subspace of dimension at least* $d-\frac{3}{2}t$*. If* $\dim(c\cap P) > \dim(P)-\frac{t}{2}$ *for all elements* c *of* Y*, then* Y *has at most* b_1^0 *elements.*

(b) Let U *be a generator,* P *a subspace of* U *of dimension* $d-\frac{3}{2}t$*, and* A *a subspace of* P *with* $\dim(A) \geqslant d-2t+1$*. If all elements* c *of* Y *satisfy* $\dim(U\cap c) = d-\frac{t}{2}$*,* $\dim(c\cap P) = d-2t$*, and* $\dim(c\cap A) \geqslant \dim(A)-\frac{t}{2}+1$*, then* Y *has at most* b_2^0 *elements.*

Proof. (a) By replacing P if necessary by a subspace of P of dimension $d-\frac{3}{2}t$, we may assume that $\dim(P) = d-\frac{3}{2}t$. Then the Gaussian coefficient in the definition of b_1^0 is the number of subspaces U of P of codimension $\frac{t}{2}-1$. By hypothesis, every element of Y contains one such subspace U. The elements of Y on such a fixed subspace U form a $(2t-1,t)$-EKR set in the quotient geometry of U.

(b) By replacing A if necessary by a subspace of A of dimension $d-2t+1$, we may assume that $\dim(A) = d-2t+1$. There are ψ^0 subspaces T of U with $\dim(T) = d-\frac{t}{2}$, $\dim(T\cap P) = d-2t$, and $\dim(T\cap A) = \dim(A)-\frac{t}{2}+1+i$ with $i \in \{0,\dots,t/2-2\}$. For each such T consider the quotient geometry $T^\perp/T$ which is isomorphic to a polar space of the same type with rank $\frac{t}{2}$. It is well-known (see for example Corollary 1.24) that there are exactly $q^{e\frac{t}{2}+\binom{t/2}{2}}$ generators in $T^\perp/T$ disjoint to U/T. Hence, there are exactly $q^{e\frac{t}{2}+\binom{t/2}{2}}$ generators a with $a \cap U = T$.

□

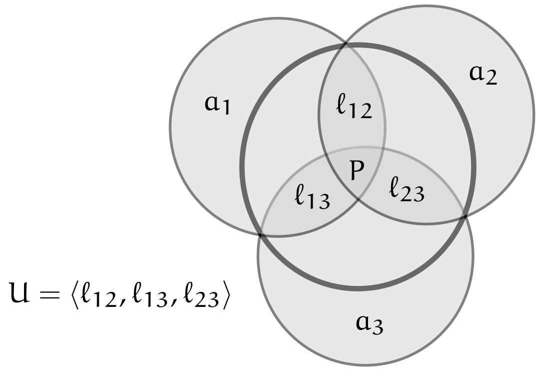

Figure 1: The setting of Lemma 5.8.

We write P_{ijk} for $a_i \cap a_j \cap a_k$, ℓ_{ij} for $a_i \cap a_j$ and U_{ijk} for $\langle a_i \cap a_j, a_i \cap a_k, a_j \cap a_k \rangle$ in the remaining parts of this chapter. Hereby we are allowed to substitute i, j, or k with other symbols. This is a purely formal convention. The strings P_{ijk} and U_{ijk} are only an expression if a_i, a_j, and a_k are appropriately defined.

Lemma 5.8. *Let Y be a (d, t)-EKR set, and consider $a_1, a_2, a_3 \in Y$. Then the following holds true.*

(a) The dimension of $a_1 \cap a_2 \cap a_3$ is at least $d - \frac{3}{2}t$.

(b) Suppose that equality holds in Part (a) and put $U := U_{123}$ and $P := a_1 \cap a_2 \cap a_3$. Then

  (i) $\dim(U) = d$.

  (ii) $\dim(\ell_{ij}) = d - t$ for $1 \leqslant i < j \leqslant 3$.

  (iii) $\dim(a_i \cap U) = d - \frac{t}{2}$ and $a_i \cap U = \langle \ell_{ij}, \ell_{ik} \rangle$ for $\{i, j, k\} = \{1, 2, 3\}$.

(iv) *Every* $b \in Y$ *satisfies* $\dim(b \cap U) \geqslant d - \frac{t}{2}$ *or* $\dim(b \cap P) > d - 2t$.

Proof. (a) As Y is a (d,t)-EKR set, then $\dim(\ell_{ij}) \geqslant d - t$. As U is totally isotropic, then $\dim(U) \leqslant d$. Since $\ell_{12} \cap \ell_{23} = P$ as well as $\langle \ell_{12}, \ell_{13} \rangle \cap \ell_{23} = P$ (because $\langle \ell_{12}, \ell_{13} \rangle \cap \ell_{23} \subseteq a_1 \cap \ell_{23} = P$), then

$$\begin{aligned} d \geqslant \dim(U) &= \dim(\langle \ell_{12}, \ell_{13}, \ell_{23} \rangle) \\ &= \dim(\langle \ell_{12}, \ell_{13} \rangle) + \dim(\ell_{23}) - \dim(P) \\ &= \dim(\ell_{12}) + \dim(\ell_{13}) - \dim(P) + \dim(\ell_{23}) - \dim(P) \\ &\geqslant 3(d-t) - 2\dim(P). \end{aligned}$$

Hence, $\dim(P) \geqslant d - \frac{3}{2}t$.

(b) As $\dim(P) = d - \frac{3}{2}t$, the argument in (a) shows that $\dim(U) = d$ and $\dim(\ell_{ij}) = d - t$ for all i, j. For $\{i,j,k\} = \{1,2,3\}$, we have $\ell_{ij}, \ell_{ik} \subseteq a_i \cap U$, and so $U = \langle a_i \cap U, \ell_{jk} \rangle$. Also $a_i \cap U \cap \ell_{jk} = a_i \cap \ell_{jk} = P$, and hence

$$\begin{aligned} d = \dim(U) &= \dim(\langle a_i \cap U, \ell_{jk} \rangle) \\ &= \dim(a_i \cap U) + \dim(\ell_{jk}) - \dim(P) \\ &= \dim(a_i \cap U) + (d-t) - (d - \frac{3}{2}t). \end{aligned}$$

Therefore, $\dim(a_i \cap U) = d - \frac{t}{2}$. Hence, as

$$\dim(\langle \ell_{ij}, \ell_{ik} \rangle) = \dim(\ell_{ij}) + \dim(\ell_{ik}) - \dim(P) = d - \frac{t}{2},$$

we have $a_i \cap U = \langle \ell_{ij}, \ell_{ik} \rangle$. We have proved the first three statements. For the final part, consider $b \in Y$.

Part (a) shows that $\dim(b \cap \ell_{ij}) \geqslant d - \frac{3}{2}t$. Put $\ell'_{ij} := b \cap \ell_{ij}$. Then $\langle \ell'_{12}, \ell'_{13} \rangle \cap \ell'_{23} \subseteq (a_1 \cap b) \cap \ell_{23} = b \cap P$ and hence equality holds. Clearly, $\ell'_{12} \cap \ell'_{13} = b \cap P$. This implies that

$$\begin{aligned} \dim(b \cap U) &\geqslant \dim(\langle \ell'_{12}, \ell'_{13}, \ell'_{23} \rangle) \\ &\geqslant \dim(\ell'_{12}) + \dim(\ell'_{13}) + \dim(\ell'_{23}) - 2\dim(b \cap P) \\ &\geqslant 3(d - \frac{3}{2}t) - 2\dim(b \cap P). \end{aligned}$$

Hence, if $\dim(b \cap P) \leqslant d - 2t$, then $\dim(b \cap U) \geqslant d - \frac{t}{2}$.

□

Lemma 5.9. *Let $t > 0$. If $\dim(a_1 \cap a_2 \cap a_3) > d - \frac{3}{2}t$ for all a_1, a_2, a_3 of a maximal (d,t)-EKR set Y, then $|Y| \leqslant b_1^0$.*

Proof. By Lemma 5.4, there exist $a_1, a_2 \in Y$ with $\dim(a_1 \cap a_2) = d - t$. Consider a third element $a_3 \in Y$ and put $P := a_1 \cap a_2 \cap a_3$. Consider any element $b \in Y$. By hypothesis, $\dim(b \cap a_1 \cap a_2) \geqslant d - \frac{3}{2}t + 1$. As P and $b \cap a_1 \cap a_2$ lie in $a_1 \cap a_2$, the dimension formula shows that

$$\begin{aligned} \dim(b \cap P) &= \dim((b \cap a_1 \cap a_2) \cap P) \\ &\geqslant \dim(b \cap a_1 \cap a_2) + \dim(P) - \dim(a_1 \cap a_2) \\ &\geqslant \dim(P) + 1 - \frac{t}{2}. \end{aligned}$$

Lemma 5.7 shows that $|y| \leqslant b_1^0$. □

Lemma 5.10. *Let Y be a (d,t)-EKR set such that there exists a generator U and a $(d - \frac{3}{2}t)$-space $P \subseteq U$ such that $a \in Y$ implies that*

$$\dim(a \cap U) \geqslant d - \frac{t}{2} \text{ or } \dim(a \cap P) > d - 2t.$$

If $\dim(a \cap U) < d - \frac{t}{2}$ for at least one element a of Y, then $|Y| \leqslant b_1^0 + b_2^0$.

Proof. For $a \in Y$ with $\dim(a \cap U) \geqslant d - \frac{t}{2}$ we have

$$\begin{aligned} \dim(a \cap P) &= \dim(a \cap U) + \dim(P) - \dim(\langle a \cap U, P \rangle) \\ &\geqslant \dim(a \cap U) + \dim(P) - \dim(U) \geqslant d - 2t \end{aligned}$$

with equality only if $\dim(a \cap U) = d - \frac{t}{2}$. Hence, with

$$\begin{aligned} Y_1 &= \{a \in Y : \dim(a \cap U) = d - \frac{t}{2}, \dim(a \cap P) = d - 2t\}, \\ Y_2 &= \{a \in Y : \dim(a \cap P) > d - 2t\}, \end{aligned}$$

we have a partition $Y = Y_1 \cup Y_2$ of Y. The first part of Lemma 5.7 gives $|Y_2| \leqslant b_1^0$. By hypothesis, there exists a element $a_2 \in Y$ with $\dim(a_2 \cap U) < d - \frac{t}{2}$. Then $a_2 \cap P$ has dimension at least $d - 2t + 1$. We shall show that $\dim(a_1 \cap a_2 \cap P) > \dim(a_2 \cap P) - \frac{t}{2}$ for all $a_1 \in Y_1$.

Then the second part of Lemma 5.7 with $A := a_2 \cap P$ gives $|Y_1| \leqslant b_2^0$ and we are done. Consider any element $a_1 \in Y_1$.

We want to show that $\langle a_1 \cap P, a_2 \cap P \rangle$ is a proper subspace of P. Suppose to the contrary that $\langle a_1 \cap P, a_2 \cap P \rangle = P$. As $\dim(U) - \dim(a_1 \cap U) = t/2 = \dim(P) - \dim(a_1 \cap P)$, this implies that $U = \langle a_1 \cap U, a_2 \cap P \rangle$. Hence, every point of $a_1 \cap a_2$ lies in $U^\perp$; but U is a generator, so $a_1 \cap a_2 \subseteq U$. It follows that

$$\begin{aligned}
\dim(a_2 \cap U) \geqslant \dim(\langle a_1 \cap a_2, a_2 \cap P \rangle) \\
&= \dim(a_1 \cap a_2) + \dim(a_2 \cap P) - \dim(a_1 \cap a_2 \cap P) \\
&= \dim(a_1 \cap a_2) - \dim(a_1 \cap P) \\
&\qquad\qquad + \dim(\langle a_1 \cap P, a_2 \cap P \rangle) \\
&= \dim(a_1 \cap a_2) + \dim(P) - \dim(a_1 \cap P) \\
&= \dim(a_1 \cap a_2) + \frac{t}{2} \geqslant d - \frac{t}{2}.
\end{aligned}$$

Here we use $\dim(a_1 \cap a_2) \geqslant d - t$, since $a_1, a_2 \in Y$. This contradicts $\dim(a_2 \cap U) < d - \frac{t}{2}$.

Hence $\langle a_1 \cap P, a_2 \cap P \rangle$ is a proper subspace of P and thus has dimension at most $\dim(P) - 1 = d - \frac{3}{2}t - 1$. It follows that

$$\begin{aligned}
\dim(a_1 \cap a_2 \cap P) &= \dim((a_1 \cap P) \cap (a_2 \cap P)) \\
&\geqslant \dim(a_1 \cap P) + \dim(a_2 \cap P) - (d - \frac{3}{2}t - 1) \\
&= \dim(a_2 \cap P) - \frac{t}{2} + 1.
\end{aligned}$$

This completes the proof. □

Example 5.11. *A d-junta, i.e. the set consisting of all generators that meet a given generator in a subspace of dimension at least $d - \frac{t}{2}$, is a maximal (d, t)-EKR set.*

Proof. As the given generator U has dimension d, the dimension formula shows that the meet of two elements of Y has dimension at least $d - t$, thus Y is a (d, t)-EKR set. Consider any generator T with $\dim(U \cap T) < d - \frac{t}{2}$. Then U has a subspace R of dimension $d - \frac{t}{2}$ such that $\dim(R \cap U \cap T) < d - t$. The subspace $T' := \langle R, R^\perp \cap T \rangle$ is a generator on R and in the quotient on R one sees that there exists a generator T'' on R with $T'' \cap T' = R$. Then

$$\begin{aligned} T'' \cap T &= T'' \cap T \cap R^{\perp} \\ &= T'' \cap T \cap T' = T \cap R. \end{aligned}$$

Hence, $\dim(T \cap T'') < d - t$. As $T'' \in Y$, this shows that $Y \cup \{T\}$ is not a (d,t)-EKR set and hence Y is maximal. □

Theorem 5.12. *Let Y be a maximal (d,t)-EKR set with $|Y| > b_1^0 + b_2^0$. Then Y is a d-junta.*

Proof. In the view of Lemma 5.9 and Lemma 5.8 (a) there are distinct elements $a_1, a_2, a_3 \in Y$ such that $P := a_1 \cap a_2 \cap a_3$ has dimension $d - \frac{3}{2}t$. Put $U := \langle a_1 \cap a_2, a_1 \cap a_3, a_2 \cap a_3 \rangle$. Lemma 5.8 gives $\dim(U) = d$ and shows that every $b \in Y$ satisfies $\dim(b \cap U) \geqslant d - \frac{t}{2}$ or $\dim(b \cap P) > d - 2t$. If $\dim(b \cap U) \geqslant d - \frac{t}{2}$ for all $b \in Y$, then the maximality of Y implies that Y is as in Example 5.11. Otherwise, Lemma 5.10 shows that $|Y| \leqslant b_1^0 + b_2^0$. □

The following result was already shown by Brouwer and Hemmeter in [14] for $e \neq \frac{1}{2}, \frac{3}{2}$.

Corollary 5.13. *Let Y be a maximal $(d,2)$-EKR set with $d > 2$. Then either Y is as in Example 5.11 or all elements of Y contain a fixed $(d-3)$-space.*

Proof. In this case $b_2^0 = 0$, since

$$\psi^0 = 0.$$

Hence, in the proof of Lemma 5.10 $|Y_1| = 0$. Recall that P in the proof of Lemma 5.10 has dimension $d - \frac{3}{2}t = d - 3$ and that every subspace of Y_2 meets P in a subspace of at least dimension $d - 2t + 1 = d - 3$, hence contains P. Therefore either all elements of Y contain a fixed $(d-3)$-space P or Y is as in Example 5.11 by the proof of Theorem 5.12. □

5.4 EKR SETS, t ODD

Throughout this section we assume that we work in a finite classical polar space of rank $d > 2$ and given type e. Also Y is a maximal (d,t)-EKR set with t odd.

Definition 5.14. *Define the constants* b_1^1, b_2^1, b_3^1 *by*

$$\begin{aligned} b_1^1 =& \begin{bmatrix} d-\frac{3}{2}t+\frac{1}{2} \\ (t-3)/2 \end{bmatrix} c_{2t-2,t}, \\ b_2^1 =& \omega(d,(t+1)/2)\psi^1, \\ b_3^1 =& q^{\frac{t-1}{2}e+\binom{(t-1)/2}{2}}\overline{\psi}^1. \end{aligned}$$

Lemma 5.15. *Let* Y *be a* (d,t)*-EKR set.*

(a) Let P *be a totally isotropic subspace of dimension at least* $d-\frac{3}{2}t+\frac{1}{2}$*. If* $\dim(c \cap P) \geqslant \dim(P) - \frac{t}{2} + \frac{3}{2}$ *for all elements* $c \in Y$*, then* $|Y| \leqslant b_1^1$.

(b) Let U *be a totally isotropic subspace of dimension* $d-1$*,* P *a subspace of* U *of dimension* $d-\frac{3}{2}t+\frac{1}{2}$*, and* A *a subspace of* P *with* $\dim(A) \geqslant d-2t+2$*. If all elements* $c \in Y$ *satisfy* $\dim(U \cap c) = d-\frac{t}{2}-\frac{1}{2}$*,* $\dim(c \cap P) = d-2t+1$*, and* $\dim(c \cap A) \geqslant \dim(A) - \frac{t}{2} + \frac{3}{2}$*, then* $|Y| \leqslant b_2^1$.

(c) Let G *be a generator, and* P *a subspace of* G *of dimension* $d-\frac{3}{2}t+\frac{1}{2}$*. If all* $c \in Y$ *satisfy* $\dim(G \cap c) = d-\frac{t}{2}+\frac{1}{2}$ *and* $\dim(c \cap P) = d-2t+1$*, then* $|Y| \leqslant b_3^1$.

Proof. (a) By replacing P if necessary by a subspace of dimension $d-\frac{3}{2}t+\frac{1}{2}$, we may assume that $\dim(P) = d-\frac{3}{2}t+\frac{1}{2}$. Then the Gaussian coefficient in the definition of b_1^1 is the number of subspaces U of P of codimension $\frac{t}{2}-\frac{3}{2}$. By hypothesis, every element of Y contains one such subspace U. The elements of Y on such a fixed subspace U form a $(2t-2,t)$-EKR set in the quotient geometry on U.

(b) By replacing A if necessary by a subspace of A of dimension $d-2t+2$, we may assume that $\dim(A) = d-2t+2$. There are ψ^1 subspaces T of U with $\dim(T) = d-\frac{t}{2}-\frac{1}{2}$, $\dim(T \cap P) = d-2t+$

1, and $\dim(T \cap A) = \dim(A) - \frac{t}{2} + \frac{3}{2} + i$ with $i \in \{0, \dots, t/2 - 5/2\}$. For each such T, there are exactly $\omega(d, (t+1)/2)$ generators a with $a \cap U \supseteq T$.

(c) There are exactly $\overline{\psi}^1$ subspaces T of G with $\dim(T) = d - \frac{t}{2} + \frac{1}{2}$ and $\dim(T \cap P) = d - 2t + 1$. For each such T, there are exactly $q^{\frac{t-1}{2}e + \binom{(t-1)/2}{2}}$ generators a with $T = a \cap G$, since it is well-known that $q^{\frac{t-1}{2}e + \binom{(t-1)/2}{2}}$ generators are disjoint to G in the quotient geometry of T (see for example Corollary 1.24 with $d = (t-1)/2$ and $s = d$).

□

Lemma 5.16. *Let Y be a (d, t)-EKR set, and consider $a_1, a_2, a_3 \in Y$. Then the following holds true.*

(a) The dimension of $a_1 \cap a_2 \cap a_3$ is at least $d - \frac{3}{2}t + \frac{1}{2}$.

(b) Suppose that equality holds in (a) and put $\ell_{ij} = a_i \cap a_j$ for different $i, j \in \{1, 2, 3\}$, $U := \langle \ell_{12}, \ell_{13}, \ell_{23} \rangle$, and $P = a_1 \cap a_2 \cap a_3$. Then one of the following cases occurs:

1. *(i)* $\dim(U) = d - 1$.

   (ii) $\dim(\ell_{ij}) = d - t$ *for* $1 \leqslant i < j \leqslant 3$.

   (iii) $a_i \cap U = \langle \ell_{ij}, \ell_{ik} \rangle$ *for* $\{i, j, k\} = \{1, 2, 3\}$.

   (iv) $\dim(a_i \cap U) = d - \frac{t}{2} - \frac{1}{2}$ *for* $i \in \{1, 2, 3\}$.

   (v) Every $b \in Y$ *satisfies* $\dim(b \cap P) \geqslant d - 2t + 1$ *and equality implies that* $\dim(b \cap U) \geqslant d - \frac{t}{2} - \frac{1}{2}$.

2. *(i)* $\dim(U) = d$.

   (ii) $\dim(\ell_{ij}) = \dim(\ell_{ik}) = d - t$ *and* $\dim(\ell_{jk}) = d - t + 1$ *for some* $\{i, j, k\} = \{1, 2, 3\}$.

   (iii) $a_i \cap U = \langle \ell_{ij}, \ell_{ik} \rangle$ *for* $\{i, j, k\} = \{1, 2, 3\}$.

   (iv) $\dim(a_j \cap U) = \dim(a_k \cap U) = d - \frac{t}{2} + \frac{1}{2}$, *and* $\dim(a_i \cap U) = d - \frac{t}{2} - \frac{1}{2}$ *for some* $\{i, j, k\} = \{1, 2, 3\}$ *(with the same order as in (ii)).*

(v) *Every* $b \in Y$ *satisfies* $\dim(b \cap P) \geqslant d - 2t + 1$ *and equality implies that* $\dim(b \cap U) \geqslant d - \frac{t}{2} - \frac{1}{2}$. *Also, if* $\dim(b \cap P) = d - 2t + 1$ *and* $\dim(a_i \cap a_j) = d - t$, *then* $\dim(a_i \cap a_j \cap b) = d - \frac{3}{2}t + \frac{1}{2}$.

Proof. (a) As Y is a (d,t)-EKR set, then $\dim(\ell_{ij}) \geqslant d - t$. As U is totally isotropic, then $\dim(U) \leqslant d$. Since $\ell_{12} \cap \ell_{13} = P$ as well as $\langle \ell_{12}, \ell_{13} \rangle \cap \ell_{23} = P$ (because $\langle \ell_{12}, \ell_{13} \rangle \cap \ell_{23} \subseteq a_1 \cap \ell_{23} = P$), then

$$
\begin{aligned}
d \geqslant \dim(U) &= \dim(\langle \ell_{12}, \ell_{13}, \ell_{23} \rangle) \\
&= \dim(\langle \ell_{12}, \ell_{23} \rangle) + \dim(\ell_{23}) - \dim(P) \\
&= \dim(\ell_{12}) + \dim(\ell_{13}) - \dim(P) + \dim(\ell_{23}) - \dim(P) \\
&\geqslant 3(d-t) - 2\dim(P).
\end{aligned}
$$

Hence, $\dim(P) \geqslant d - \frac{3}{2}t + \frac{1}{2}$.

(b) As $\dim(P) = d - \frac{3}{2}t + \frac{1}{2}$, the argument in (a) shows that $\dim(U) \in \{d-1, d\}$.

For $\{i,j,k\} = \{1,2,3\}$, we have $\ell_{ij}, \ell_{ik} \subseteq a_i \cap U$ and hence $U = \langle a_i \cap U, \ell_{jk} \rangle$. Also $a_i \cap U \cap \ell_{jk} = a_i \cap \ell_{jk} = P$. Hence,

$$
\begin{aligned}
\dim(U) &= \dim(\langle a_i \cap U, \ell_{jk} \rangle) \\
&= \dim(a_i \cap U) + \dim(\ell_{jk}) - \dim(P) \\
&= \dim(a_i \cap U) + \dim(\ell_{jk}) - (d - \frac{3}{2}t + \frac{1}{2}).
\end{aligned}
$$

Consider first the case that $\dim(U) = d - 1$. Then the argument to prove the first part of the lemma yields $\dim(\ell_{ij}) = d - t$ for all $i \neq j$, $i,j \in \{1,2,3\}$. Therefore, $\dim(a_i \cap U) = d - \frac{t}{2} - \frac{1}{2}$, which implies that $a_i \cap U = \langle \ell_{ij}, \ell_{ik} \rangle$.

Now consider the case that $\dim(U) = d$. Then the argument to prove the first part of the lemma yields $\dim(\ell_{ij}) = \dim(\ell_{ik}) = d - t$ and $\dim(\ell_{jk}) = d - t + 1$ for some $\{i,j,k\} = \{1,2,3\}$. Therefore, with the same choice of i,j,k, $\dim(a_i \cap U) = d - \frac{t}{2} - \frac{1}{2}$ and $\dim(a_j \cap U) = d - \frac{t}{2} + \frac{1}{2} = \dim(a_k \cap U)$. This implies that $a_i \cap U = \langle \ell_{ij}, \ell_{ik} \rangle$ for all $\{i,j,k\} = \{1,2,3\}$.

The final part is proved for both cases together.

Consider $b \in Y$. We may assume that $\dim(a_1 \cap a_2) = d - t$. It follows from the first statement of the lemma that $\ell'_{ij} := b \cap \ell_{ij}$, $1 \leqslant i < j \leqslant 3$ has dimension at least $d - \frac{3}{2}t + \frac{1}{2}$. As P and $\ell'_{12} = b \cap a_1 \cap a_2$ lie in $a_1 \cap a_2$, the dimension formula shows that

$$\begin{aligned}\dim(b \cap P) &= \dim(\ell'_{12} \cap P)\\ &\geqslant \dim(\ell'_{12}) + \dim(P) - \dim(a_1 \cap a_2)\\ &\geqslant 2(d - \frac{3}{2}t + \frac{1}{2}) - (d - t) = d - 2t + 1,\end{aligned}$$

and equality implies that $\ell'_{12} = a_1 \cap a_2 \cap b$ has dimension $d - \frac{3}{2}t + \frac{1}{2}$. Suppose finally that $\dim(b \cap P) = d - 2t + 1$. We have $\langle \ell'_{12}, \ell'_{13} \rangle \cap \ell'_{23} \subseteq (a_1 \cap b) \cap \ell_{23} = b \cap P$ and $\ell'_{12} \cap \ell'_{13} = b \cap P$. This implies that

$$\begin{aligned}\dim(b \cap U) &\geqslant \dim(\langle \ell'_{12}, \ell'_{13}, \ell'_{23} \rangle)\\ &\geqslant \dim(\ell'_{12}) + \dim(\ell'_{13}) + \dim(\ell'_{23}) - 2\dim(b \cap P)\\ &\geqslant 3(d - \frac{3}{2}t + \frac{1}{2}) - 2\dim(b \cap P) = d - \frac{t}{2} - \frac{1}{2}.\end{aligned}$$

□

Lemma 5.17. *If* $\dim(a_1 \cap a_2 \cap a_3) > d - \frac{3}{2}t + \frac{1}{2}$ *for all* a_1, a_2, a_3 *of a maximal* (d, t)*-EKR set* Y*, then* $|Y| \leqslant b_1^1$.

Proof. Lemma 5.4 gives $a_1, a_2 \in Y$ with $\dim(a_1 \cap a_2) = d - t$. Consider a third element $a_3 \in Y$ and put $P := a_1 \cap a_2 \cap a_3$. Consider any element $b \in Y$. By hypothesis, $\dim(b \cap a_1 \cap a_2) \geqslant d - \frac{3}{2}t + \frac{3}{2}$. As P and $b \cap a_1 \cap a_2$ lie in $a_1 \cap a_2$, the dimension formula shows that

$$\begin{aligned}\dim(b \cap P) &= \dim((b \cap a_1 \cap a_2) \cap P)\\ &\geqslant \dim(b \cap a_1 \cap a_2) + \dim(P) - \dim(a_1 \cap a_2)\\ &\geqslant \dim(P) - \frac{t}{2} + \frac{3}{2}.\end{aligned}$$

As $\dim(P) > d - \frac{3}{2}t - \frac{1}{2}$ (by hypothesis), Lemma 5.15 proves the assertion. □

Lemma 5.18. *Let Y be a (d,t)-EKR set such that there exist a generator G_0, a $(d-1)$-space $U \subseteq G_0$, and $(d-\frac{3}{2}t+\frac{1}{2})$-spaces $P, Q \subseteq U$ such that $a \in Y$ implies the following:*

$$\dim(a \cap U) \geqslant d - \frac{t}{2} - \frac{1}{2} \text{ or}$$
$$\dim(a \cap P) > d-2t+1 \text{ or } \dim(a \cap Q) > d-2t+1.$$

Suppose also that $\dim(a \cap U) < d - \frac{t}{2} - \frac{1}{2}$ for at least one element a of Y. Then $|Y| \leqslant 2b_1^1 + b_2^1 + b_3^1$.

Proof. For $a \in Y$ with $\dim(a \cap U) \geqslant d - \frac{t}{2} - \frac{1}{2}$ we have

$$\begin{aligned}\dim(a \cap P) &= \dim(a \cap U) + \dim(P) - \dim(\langle a \cap U, P\rangle)\\ &\geqslant \dim(a \cap U) + \dim(P) - \dim(U) \geqslant d-2t+1.\end{aligned}$$

Hence, with

$$\begin{aligned}Y_1 &:= \{a \in Y : \dim(a \cap U) = d - \frac{t}{2} - \frac{1}{2}, \dim(a \cap P) = d-2t+1\},\\ Y_2 &:= \{a \in Y : \dim(a \cap P) > d-2t+1\},\\ Y_3 &:= \{a \in Y : \dim(a \cap Q) > d-2t+1\},\end{aligned}$$

we have a cover $Y_1 \cup Y_2 \cup Y_3$ of Y. Lemma 5.15 gives $|Y_2| + |Y_3| \leqslant 2b_1^1$.

By hypothesis, there exists a element $a_2 \in Y_2$ with $\dim(a_2 \cap U) \leqslant d - \frac{t}{2} - \frac{3}{2}$. Define the following two subsets of Y_1:

$$\begin{aligned}S &:= \{a_1 \in Y_1 : \langle a_1 \cap P, a_2 \cap P\rangle = P\}\\ T &:= \{a_1 \in Y_1 : \langle a_1 \cap P, a_2 \cap P\rangle \neq P\}.\end{aligned}$$

In the following, we use Lemma 5.15 to show that $|S| \leqslant b_3^1$ and $|T| \leqslant b_2^1$.

Let $a_1 \in S$. As $\dim(U) - \dim(a_1 \cap U) = \frac{t}{2} - \frac{1}{2} = \dim(P) - \dim(a_1 \cap P)$ we have $U = \langle a_1 \cap U, P\rangle$. As $P = \langle a_1 \cap P, a_2 \cap P\rangle$ this implies that $U = \langle a_1 \cap U, a_2 \cap P\rangle$ and hence $U = \langle a_1 \cap U, a_2 \cap U\rangle$. Therefore every point of $a_1 \cap a_2$ lies in $U^\perp$ and

$$\begin{aligned}\dim(a_1 \cap a_2 \cap U) &= \dim(a_1 \cap U) + \dim(a_2 \cap U) - \dim(U)\\ &\leqslant (d - \frac{t}{2} - \frac{1}{2}) + (d - \frac{t}{2} - \frac{3}{2}) - (d-1)\\ &= d-t-1.\end{aligned}$$

As $a_1, a_2 \in Y$, we have $\dim(a_1 \cap a_2) \geqslant d-t$. As U has dimension $d-1$, it follows that U and $a_1 \cap a_2$ span a generator G, which implies that $\dim(a_1 \cap a_2) = d-t$ and $\dim(a_1 \cap a_2 \cap U) = d-t-1$, which in turn shows that $\dim(a_2 \cap U) = d - \frac{t}{2} - \frac{3}{2}$. Then $\dim(a_1 \cap G) = \dim(a_1 \cap U) + 1 = d - \frac{t}{2} + \frac{1}{2}$. Clearly, $G = \langle U, U^{\perp} \cap a_2 \rangle$ and thus G is independent of the choice of $a_1 \in S$. Hence every element of S meets G in a subspace of dimension $d - \frac{t}{2} + \frac{1}{2}$. Recall $S \subseteq Y_1$, so $\dim(a_1 \cap P) = d - 2t + 1$. Applying the third part of Lemma 5.15 now gives

$$|S| \leqslant b_3^1.$$

For $a_1 \in T$ we have that $\langle a_1 \cap P, a_2 \cap P \rangle$ is a proper subspace of P , so we have

$$\begin{aligned}
&\dim(a_1 \cap a_2 \cap P) \\
&\geqslant \dim(a_1 \cap P) + \dim(a_2 \cap P) - \dim(\langle a_1 \cap P, a_2 \cap P \rangle) \\
&\geqslant \dim(a_1 \cap P) + \dim(a_2 \cap P) - \dim(P) + 1 \\
&= \dim(a_2 \cap P) - \frac{t}{2} + \frac{3}{2}.
\end{aligned}$$

Then the second part of Lemma 5.15 applied with $A = a_2 \cap P$ gives $|T| \leqslant b_2^1$. Hence $|Y| = |Y_2| + |Y_3| + |S| + |T| < 2b_1^1 + b_2^1 + b_3^1$. $\square$

Example 5.19. *A $(d-1,t)$-junta, i.e. the set consisting of all generators which meet a given $(d-1)$-space U in a subspace of dimension at least $d - \frac{t}{2} - \frac{1}{2}$, is a maximal (d,t)-EKR set.*

Proof. As the given subspace U has dimension $d-1$, the dimension formula shows that the meet any of two elements of Y has dimension at least $d-t$, thus Y is a (d,t)-EKR set. Consider any generator T with $\dim(U \cap T) < d - \frac{t}{2} - \frac{1}{2}$. Then G has a subspace R of dimension $d - \frac{t}{2} - \frac{1}{2}$ such that $\dim(R \cap G \cap T) < d-t$. The subspace $T' := \langle R, R^{\perp} \cap T \rangle$ is a generator on R and in the quotient on R one sees that there exists a generator T'' on R with $T'' \cap T' = R$. Then $T'' \cap T = R \cap G \cap T$ and hence $\dim(T \cap T'') < d-t$. As $T'' \in Y$, this shows that $Y \cup \{T\}$ is not a (d,t)-EKR set. $\square$

Lemma 5.20. *Let Y be a (d,t)-EKR set. Let $a_1, a_2, a_3, a_4 \in Y$. Suppose that we have*

$$\dim(P_{123}) = \dim(P_{124}) = d - \frac{3}{2}t + \frac{1}{2},$$

$\dim(a_4 \cap P_{123}) = d - 2t + 1$, *and* $\dim(a_1 \cap a_2) = d - t$. *Let* $U = U_{123} \cap U_{124}$. *Then*

$$\dim(U) \geqslant d - 1 \text{ and } \dim(a_4 \cap U) \geqslant \dim(U) - \frac{t}{2} + \frac{1}{2}.$$

Proof. By Lemma 5.16 (a), $\dim(P_{ijk}) \geqslant d - \frac{3}{2}t + \frac{1}{2}$. Hence,

$$\begin{aligned}
\dim(U) &\geqslant \dim(\langle a_1 \cap a_2, a_1 \cap a_3 \cap a_4, a_2 \cap a_3 \cap a_4 \rangle) \\
&\geqslant \dim(a_1 \cap a_2) + \dim(\langle a_1 \cap a_3 \cap a_4, a_2 \cap a_3 \cap a_4 \rangle) \\
&\qquad\qquad - \dim(a_4 \cap P_{123}) \\
&= (d - t) + \dim(a_1 \cap a_3 \cap a_4) + \dim(a_2 \cap a_3 \cap a_4) \\
&\qquad\qquad - 2\dim(a_4 \cap P_{123}) \\
&\geqslant (d - t) + 2(d - \frac{3}{2}t + \frac{1}{2}) - 2\dim(a_4 \cap P_{123}) = d - 1.
\end{aligned}$$

This shows the first part of the assertion.

If $\dim(U_{123}) = \dim(U_{124}) = d - 1$, then the claim on $\dim(a_4 \cap U)$ follows by Lemma 5.16 (b) 1.(iv). Hence suppose without loss of generality $\dim(U_{124}) = d$. Then Lemma 5.16 (b) 2.(ii) shows that $\dim(a_1 \cap a_4) + \dim(a_2 \cap a_4) = 2d - 2t + 1$. Then, by Lemma 5.16 (b) 2.(iv), $\dim(a_4 \cap U_{124}) = d - \frac{t}{2} + \frac{1}{2}$. Hence, $\dim(a_4 \cap U) \geqslant d - \frac{t}{2} - \frac{1}{2}$. □

Definition 5.21. *We call a (d,t)-EKR set Y* funny *if it has the following property: For all $a_1, a_2, a_3 \in Y$ we have that*

$$\dim(a_1 \cap a_2 \cap a_3) = d - \frac{3}{2}t + \frac{1}{2}$$

implies $\dim(U_{123}) = d$.

Remark 5.22. *A $(d-1,t)$-junta is funny if and only if $e = 0$.*

Lemma 5.23. *If a maximal (d,t)-EKR set Y is funny, but not a $(d-1,t)$-junta, then $|Y| \leqslant 2b_1^1 + b_2^1 + b_3^1$.*

Proof. Let Y_0 be a $(d-1,t)$-junta. By assumption, $Y \neq Y_0$. Furthermore, Y is maximal, so Y can not be a subset of Y_0 either. We shall use this in the proof.

If all $a_1, a_2, a_3 \in Y$ satisfy $\dim(a_1 \cap a_2 \cap a_3) > d - \frac{3}{2}t + \frac{1}{2}$, then Lemma 5.17 shows $|Y| \leqslant b_1^1$. Hence suppose that there are $a_1, a_2, a_3 \in Y$ with $\dim(a_1 \cap a_2 \cap a_3) = d - \frac{3}{2}t + \frac{1}{2}$, $\dim(a_1 \cap a_2) = d - t$, and $\dim(a_1 \cap U_{123}) = d - \frac{t}{2} - \frac{1}{2}$ (see Lemma 5.16 (b) 2.). Set

$$Y_1 := \{a \in Y : \dim(a \cap P_{123}) > d - 2t + 1\},$$
$$Y_2 := \{a \in Y : \dim(a \cap P_{123}) = d - 2t + 1\}.$$

By Lemma 5.16 (b) 2.(v), $Y = Y_1 \cup Y_2$ is a partition of Y. We have $|Y| \leqslant b_1^1$ by Lemma 5.15. We may thus assume that $Y_2 \neq \emptyset$.

Case 1. All $a_4 \in Y_2$ **satisfy** $\dim(U_{123} \cap U_{124}) = d$**.** Let $U \subseteq U_{123}$ be a $(d-1)$-dimensional subspace of U_{123} with $a_1 \cap U_{123} \subseteq U$. By Lemma 5.16 (b) 2.(v) and $\dim(a_4 \cap P_{123}) = d - 2t + 1$, $\dim(P_{124}) = d - \frac{3}{2}t + \frac{1}{2}$. By Lemma 5.16 (b) 2. (iv) and $\dim(a_1 \cap a_2) = d - t$, all $a_4 \in Y_2$ satisfy

$$\begin{aligned}\dim(a_4 \cap U) &\geqslant \dim(a_4 \cap U_{124}) - 1\\ &= d - \frac{t}{2} - \frac{1}{2}.\end{aligned}$$

As Y is not a subset of Y_0, there exists a $a_5 \in Y$ with $\dim(a_5 \cap U) < d - \frac{t}{2} - \frac{1}{2}$. Hence, we can apply Lemma 5.18 with $G_0 = U_{123}$ and $P = Q = P_{123}$. This shows $|Y| \leqslant 2b_1^1 + b_2^1 + b_3^1$.

Case 2. There exists a generator $a_4 \in Y_2$ **with** $\dim(U_{123} \cap U_{124}) \leqslant d - 1$**.** Put $U := U_{123} \cap U_{124}$. By Lemma 5.16 (b) 2.(v) and $\dim(a_4 \cap P_{123}) = d - 2t + 1$, $\dim(a_1 \cap a_2) = d - t$, and $\dim(P_{124}) = d - \frac{3}{2}t + \frac{1}{2}$. Hence, by Lemma 5.20, $\dim(U) = d - 1$. Define the following subsets of Y_2:

$$S := \{a_i \in Y_2 : \dim(a_i \cap P_{124}) = d - 2t + 1\},$$
$$T := \{a_i \in Y_2 : \dim(a_i \cap P_{124}) > d - 2t + 1\}.$$

By Lemma 5.16 (b) 2.(v), this is a partition $Y_2 = S \cup T$ of Y_2. Let $a_i \in S$. By Lemma 5.16 (b) 2.(v) and $\dim(a_i \cap P_{123}) = d - 2t + 1$, $\dim(P_{12i}) = d - \frac{3}{2}t + \frac{1}{2}$. Hence, by Lemma 5.20, $\dim(U_{12i} \cap U_{123}) \geqslant$

$d-1$ and $\dim(U_{12i}\cap U_{124}) \geqslant d-1$. Suppose for a contradiction $\dim(U_{12i}\cap U_{123}\cap U_{124}) = d-2$. Then $\dim(U_{12i}\cap U_{123}) = d-1$ and $\dim(U_{12i}\cap U_{124}) = d-1$. Hence,

$$\begin{aligned} d &\geqslant \dim(\langle U_{12i}\cap U_{123}, U_{12i}\cap U_{124}, U_{123}\cap U_{124}\rangle) \\ &= \dim(U_{12i}\cap U_{123}) + \dim(U_{12i}\cap U_{124}) + \dim(U_{123}\cap U_{124}) \\ &\quad - 2\dim(U_{12i}\cap U_{123}\cap U_{124}) \\ &= 3(d-1) - 2(d-2) = d+1. \end{aligned}$$

This is a contradiction. Hence, we have $U \subseteq U_{123}\cap U_{12i}$. Hence, by Lemma 5.20, all $a_i \in S$ satisfy

$$\dim(a_i \cap U) \geqslant d - \frac{t}{2} - \frac{1}{2}.$$

As Y is not a subset of Y_0, there exists a $a_5 \in Y$ with $\dim(a_5 \cap U) < d - \frac{t}{2} - \frac{1}{2}$. Thus we can apply Lemma 5.18 with $U \subseteq G_0 = U_{123}$, $P = P_{123}$, and $Q = P_{124}$. This shows $|Y| \leqslant 2b_1^1 + b_2^1 + b_3^1$. □

Theorem 5.24. *Let Y be a maximal (d,t)-EKR set where $|Y| > 2b_1^1 + b_2^1 + b_3^1$. Then Y is a $(d-1)$-junta.*

Proof. In view of Lemma 5.23 there exist $a_1, a_2, a_3 \in Y$ such that $\dim(P_{123}) = d - \frac{3}{2}t + \frac{1}{2}$ and $\dim(U_{123}) < d$. Thus $\dim(U) = d-1$ by Lemma 5.16. Lemma 5.16 shows that every $b \in Y$ satisfies $\dim(b \cap U) \geqslant d - \frac{t}{2} - \frac{1}{2}$ or $\dim(b \cap P) > d - 2t + 1$. If $\dim(b\cap U) \geqslant d - \frac{t}{2} - \frac{1}{2}$ for all $b \in Y$, then the maximality of Y implies that Y is as in Example 5.19. Now Lemma 5.18 (with $P = Q$) shows that $|Y| \leqslant 2b_1^1 + b_2^1 + b_3^1$. □

5.5 ASSOCIATION SCHEMES OF DUAL POLAR GRAPHS REVISITED

We are interested in the eigenvalues of $\sum_{s=0}^{a} A_{d-s}$, so we shall have to explicitly calculate these. Since the V_r are the common eigenspaces of $A_0, \ldots, A_d$, the V_r are subspaces of eigenspaces of $\sum_{s=0}^{a} A_{d-s}$. Hence, the eigenvalue of $\sum_{s=0}^{a} A_{d-s}$ on V_r is $\lambda_r^a := \sum_{s=0}^{a} P_{r,d-s}$.

Theorem 5.25. *For $a < d$ we have*

$$\lambda_0^a = \sum_{s=0}^{a} \begin{bmatrix} d \\ s \end{bmatrix} q^{\binom{d-s}{2}+(d-s)e}.$$

For $a < d$ and $r > 0$ we have

$$\lambda_r^a = (-1)^{r+a} \sum_{s=\max(a-r+1,0)}^{\min(a,d-r)} (-1)^s A(r,s,a)$$

where

$$A(r,s,a) := \begin{bmatrix} d-r \\ s \end{bmatrix} q^{\binom{d-r-s}{2}+(d-r-s)e} \begin{bmatrix} r-1 \\ a-s \end{bmatrix} q^{\binom{r-a+s}{2}}.$$

Proof. The case $r = 0$ follows directly from Theorem 1.23.

In the case $r > 0$ starting with Theorem 1.23 we see

$$\begin{aligned}
\lambda_r^a &= \textstyle\sum_{s=0}^{a} \sum_{t=\max(r+s-d,0)}^{\min(s,r)} (-1)^{r-t} \left[\begin{smallmatrix} d-r \\ s-t \end{smallmatrix}\right] \left[\begin{smallmatrix} r \\ t \end{smallmatrix}\right] q^{\binom{r-t}{2}+\binom{d-r+t-s}{2}+(d-r+t-s)e} \\
&= \textstyle\sum_{t=0}^{r} (-1)^{r-t} \left[\begin{smallmatrix} r \\ t \end{smallmatrix}\right] q^{\binom{r-t}{2}} \sum_{s=t}^{\min(a,d-r+t)} \left[\begin{smallmatrix} d-r \\ s-t \end{smallmatrix}\right] q^{\binom{d-r+t-s}{2}+(d-r+t-s)e} \\
&= \textstyle\sum_{t=0}^{r} (-1)^{r-t} \left[\begin{smallmatrix} r \\ t \end{smallmatrix}\right] q^{\binom{r-t}{2}} \sum_{s=0}^{\min(a-t,d-r)} \left[\begin{smallmatrix} d-r \\ s \end{smallmatrix}\right] q^{\binom{d-r-s}{2}+(d-r-s)e} \\
&= \textstyle\sum_{s=0}^{\min(a,d-r)} \left[\begin{smallmatrix} d-r \\ s \end{smallmatrix}\right] q^{\binom{d-r-s}{2}+(d-r-s)e} \sum_{t=0}^{\min(a-s,r)} (-1)^{r-t} \left[\begin{smallmatrix} r \\ t \end{smallmatrix}\right] q^{\binom{r-t}{2}} \\
&\overset{1.28}{=} \textstyle\sum_{s=\max(a-r,0)}^{\min(a,d-r)} \left[\begin{smallmatrix} d-r \\ s \end{smallmatrix}\right] q^{\binom{d-r-s}{2}+(d-r-s)e} \sum_{t=0}^{a-s} (-1)^{r-t} \left[\begin{smallmatrix} r \\ t \end{smallmatrix}\right] q^{\binom{r-t}{2}} \\
&\overset{1.28}{=} \textstyle\sum_{s=\max(a-r+1,0)}^{\min\{a,d-r\}} \left[\begin{smallmatrix} d-r \\ s \end{smallmatrix}\right] q^{\binom{d-r-s}{2}+(d-r-s)e} (-1)^{r+a-s} \left[\begin{smallmatrix} r-1 \\ a-s \end{smallmatrix}\right] q^{\binom{r-a+s}{2}} \\
&= (-1)^{r+a} \textstyle\sum_{s=\max(a-r+1,0)}^{\min(a,d-r)} (-1)^s \left[\begin{smallmatrix} d-r \\ s \end{smallmatrix}\right] q^{\binom{d-r-s}{2}+(d-r-s)e} \left[\begin{smallmatrix} r-1 \\ a-s \end{smallmatrix}\right] q^{\binom{r-a+s}{2}} \\
&= (-1)^{r+a} \textstyle\sum_{s=\max(a-r+1,0)}^{\min(a,d-r)} (-1)^s A(r,s,a).
\end{aligned}$$

□

Corollary 5.26. *For $a \leqslant d-1$ we have*

(a)

$$\lambda_1^a = -\begin{bmatrix} d-1 \\ a \end{bmatrix} q^{\binom{d-a-1}{2}+(d-a-1)e} = -A(1,a,a),$$

(b)

$$\lambda_d^a = (-1)^{d-a} \begin{bmatrix} d-1 \\ a \end{bmatrix} q^{\binom{d-a}{2}} = (-1)^{d+a} A(d,0,a),$$

(c)

$$\lambda_{d-1}^{a} = (-1)^{d-1+a} \left(\begin{bmatrix} d-2 \\ a \end{bmatrix} q^{\binom{d-a-1}{2}+e} - \begin{bmatrix} d-2 \\ a-1 \end{bmatrix} q^{\binom{d-a}{2}} \right).$$

5.6 HOFFMAN'S BOUND REVISITED

Hoffman's bound restricts the maximum size $c_{d,t}$ of a (d, t)-EKR set. As mentioned before, it is known that this bound is sharp for $t = d-1$ except when $e = \frac{1}{2}$ and d is odd [66]. Hoffman's bound for (d, t)-EKR sets looks as follows.

Proposition 5.27 (Proposition 1.9)**.** *The size $c_{d,t}$ of a (d, t)-EKR set is bounded by*

$$c_{d,t} \leqslant \frac{n\lambda_{\min}}{\lambda_{\min} - k},$$

where $\lambda_{\min} := \min_r \lambda_r^{d-t-1}$ is the smallest eigenvalue of the adjacency matrix $\sum_{s=t+1}^{d} A_s$, and $k = \lambda_0^{d-t-1}$ is the valency of the graph associated to this matrix.

To our knowledge the smallest eigenvalue of $\sum_{s=t+1}^{d} A_{d-s}$ was never calculated except for easy cases such as $t = d-1$, so this section is concerned about approximating $\lambda_{\min}$. Our claim is the following:

Theorem 5.28. *For $a \leqslant d-1$, and $q \geqslant 3$, the following holds:*

(a) The eigenvalue λ_1^a is the maximum of $\{|\lambda_r^a| : r = 1, \ldots, d\}$ if $e \geqslant 1$.

(b) The eigenvalue λ_d^a is the maximum of $\{|\lambda_r^a| : r = 1, \ldots, d\}$ if $e \leqslant 1$.

(c) The eigenvalue λ_1^a is the minimum of $\{\lambda_r^a : r = 1, \ldots, d\}$ if $d-a$ is even or $e \geqslant 1$.

(d) The eigenvalue λ_d^a is the minimum of $\{\lambda_r^a : r = 1, \ldots, d\}$ if $d-a$ is odd and $e \leqslant 1$.

Note that the result should also hold for $q = 2$, but our techniques are ill-suited to handle this case. We shall prove Theorem 5.28 in several steps.

Lemma 5.29. *Theorem 5.28 holds for* $a = d - 1$.

Proof. It suffices to notice from Theorem 5.25 that

$$\lambda_r^{d-1} = -1$$

for all $r \in \{1, \ldots, d\}$. □

We make the following trivial observation.

Lemma 5.30. *Theorem 5.28 holds for* $d \leqslant 2$.

Proof. By Lemma 5.29, the claim holds for $(d, a) \in \{(2, 1), (1, 0)\}$. The case $(d, a) = (2, 0)$ remains. By Corollary 5.26,

$$\lambda_1^0 = -q^e \qquad \lambda_2^0 = q.$$

The assertion follows. □

Proposition 5.31. *Let* $a \leqslant d - 2$, $q \geqslant 3$. *Then*

(a)

$$|\lambda_1^a| - |\lambda_d^a| \begin{cases} > 0 & \text{if } e > 1, \\ = 0 & \text{if } e = 1, \\ < 0 & \text{if } e < 1. \end{cases}$$

(b) If $d \geqslant 3$, *then*

$$|\lambda_1^a|, |\lambda_d^a| \geqslant |\lambda_{d-1}^a|$$

Proof. By Corollary 5.26,

$$\begin{aligned} |\lambda_1^a| - |\lambda_d^a| &= \begin{bmatrix} d-1 \\ a \end{bmatrix} q^{\binom{d-a-1}{2} + (d-a-1)e} - \begin{bmatrix} d-1 \\ a \end{bmatrix} q^{\binom{d-a}{2}} \\ &= \begin{bmatrix} d-1 \\ a \end{bmatrix} q^{\binom{d-a}{2}} \left(q^{(d-a-1)(e-1)} - 1 \right). \end{aligned}$$

As $a \leqslant d-2$, the statement in (a) follows. For (b) we calculate

$$\begin{aligned}
|\lambda_{d-1}^{a}| &= \left| \begin{bmatrix} d-2 \\ a \end{bmatrix} q^{\binom{d-a-1}{2}+e} - \begin{bmatrix} d-2 \\ a-1 \end{bmatrix} q^{\binom{d-a}{2}} \right| \\
&\overset{(1.16)}{=} \left| q^{\binom{d-a-1}{2}+e} \begin{bmatrix} d-2 \\ a \end{bmatrix} + q^{\binom{d-a-1}{2}} \left(\begin{bmatrix} d-2 \\ a \end{bmatrix} - \begin{bmatrix} d-1 \\ a \end{bmatrix} \right) \right| \\
&= q^{\binom{d-a-1}{2}} \left| (q^e+1) \begin{bmatrix} d-2 \\ a \end{bmatrix} - \begin{bmatrix} d-1 \\ a \end{bmatrix} \right| \\
&= q^{\binom{d-a-1}{2}} \begin{bmatrix} d-1 \\ a \end{bmatrix} \left| (q^e+1) \frac{q^{d-1-a}-1}{q^{d-1}-1} - 1 \right|.
\end{aligned}$$

If $a < d-2$, then by Corollary 5.26

$$|\lambda_{d-1}^{a}| \leqslant q^{\binom{d-a-1}{2}+e} \begin{bmatrix} d-1 \\ a \end{bmatrix} \leqslant |\lambda_1^a|, |\lambda_d^a|.$$

If $a = d-2$, then by $d \geqslant 3$ and Corollary 5.26

$$\begin{aligned}
|\lambda_{d-1}^{a}| &\leqslant q^{\binom{d-a-1}{2}} \begin{bmatrix} d-1 \\ a \end{bmatrix} \left| (q^2+1) \frac{q-1}{q^{d-1}-1} - 1 \right| \\
&\leqslant q^{\binom{d-a-1}{2}+1} \begin{bmatrix} d-1 \\ a \end{bmatrix} \leqslant |\lambda_1^a|, |\lambda_d^a|.
\end{aligned}$$

This shows (b). □

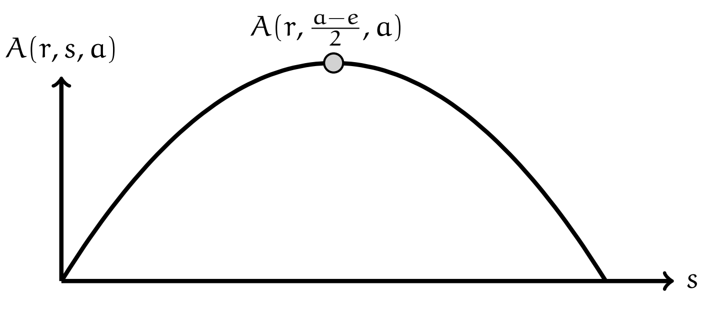

Figure 2: The function $A(r, s, a)$ imagined as a continuous unimodal function in s.

Lemma 5.32. *For fixed* $r > 0$, $q \geqslant 3$, *and* $a \in \{0, \dots, d-1\}$ *the sequence*

$$(A(r,s,a))_{\max(a-r+1,0)\leqslant s\leqslant \min(a,d-r)}$$

is unimodal. More precisely, we have

(a) If $2s + e - a \geqslant \frac{1}{2}$, *then* $A(r,s,a) > A(r,s+1,a)$.

(b) If $2s + e - a \leqslant -\frac{1}{2}$, *then* $A(r,s,a) < A(r,s+1,a)$.

Proof. We investigate the sign of $x := A(r,s,a) - A(r,s+1,a)$ for integers s with $\max(a-r+1,0) \leqslant s \leqslant \min(a,d-r) - 1$.

$$\begin{aligned} x = &\begin{bmatrix} d-r \\ s \end{bmatrix}\begin{bmatrix} r-1 \\ a-s \end{bmatrix} q^{\binom{d-r-s}{2}+\binom{r-a+s}{2}+(d-r-s)e} \\ -&\begin{bmatrix} d-r \\ s+1 \end{bmatrix}\begin{bmatrix} r-1 \\ a-s-1 \end{bmatrix} q^{\binom{d-r-s-1}{2}+\binom{r-a+s+1}{2}+(d-r-s-1)e} \end{aligned}$$

Hence, $x = q^y B$ for some integer y and

$$\begin{aligned} B := &q^{2s+e-a}\frac{\begin{bmatrix} d-r \\ s \end{bmatrix}}{q^{(d-r-s)s}} \cdot \frac{\begin{bmatrix} r-1 \\ a-s \end{bmatrix}}{q^{(r-1-a+s)(a-s)}} \\ &- \frac{\begin{bmatrix} d-r \\ s+1 \end{bmatrix}}{q^{(d-r-s-1)(s+1)}} \cdot \frac{\begin{bmatrix} r-1 \\ a-s-1 \end{bmatrix}}{q^{(r-a+s)(a-s-1)}}. \end{aligned}$$

We distinguish several cases. Notice the following implications. We shall use them without any further notice.

- If $0 = s$, then $\begin{bmatrix} d-r \\ s+1 \end{bmatrix} = [d-r]$.
- If $a - s = r - 1$, then $\begin{bmatrix} r-1 \\ a-s-1 \end{bmatrix} = [r-1]$ by (1.15).
- If $s = d - r - 1$, then $\begin{bmatrix} d-r \\ s \end{bmatrix} = [d-r]$ by (1.15).
- If $a - s - 1 = 0$, then $\begin{bmatrix} r-1 \\ a-s \end{bmatrix} = [r-1]$.

Our claim is the following.

- If $2s + e - a \geqslant \frac{1}{2}$, then $B > 0$.
- If $2s + e - a \leqslant -\frac{1}{2}$, then $B < 0$.

First we shall show the statements for $q \geqslant 4$.

Case $0 = s$ **and** $a - s = r - 1$ **and** $s = d - r - 1$ **and** $a - s - 1 = 0$**.** Obviously, $B = q^{2s+e-a} - 1$.

Case $0 = s$ **and** $a - s = r - 1$ **and (**$s = d - r - 1$ **xor** $a - s - 1 = 0$**).** By Lemma 2.2 and Lemma 2.1 (d),

$$q^{2s+e-a} - \frac{q}{q-1} \leqslant B \leqslant q^{2s+e-a} - (1 + q^{-1}).$$

Case $0 = s$ **and** $a - s = r - 1$ **and not (**$s = d - r - 1$ **or** $a - s - 1 = 0$**).** By Lemma 2.2 and Lemma 2.1 (d),

$$q^{2s+e-a} - \left(\frac{q}{q-1}\right)^2 \leqslant B \leqslant q^{2s+e-a} - (1 + q^{-1})^2.$$

Case ($0 = s$ **xor** $a - s = r - 1$**) and not (**$s = d - r - 1$ **or** $a - s - 1 = 0$**).** By Lemma 2.2, Lemma 2.1 (c), and Lemma 2.1 (d),

$$\begin{aligned} &q^{2s+e-a}(1 + q^{-1}) - (1 + 2q^{-1}) \cdot \frac{q}{q-1} \\ &\leqslant B \leqslant q^{2s+e-a}(1 + 2q^{-1}) - (1 + q^{-1})^2. \end{aligned}$$

Case $s = d - r - 1$ **and** $a - s - 1 = 0$ **and (**$0 = s$ **xor** $a - s = r - 1$**).** By Lemma 2.2 and Lemma 2.1 (d),

$$q^{2s+e-a}(1 + q^{-1}) - 1 \leqslant B \leqslant q^{2s+e-a} \cdot \frac{q}{q-1} - 1.$$

Case $s = d - r - 1$ **and** $a - s - 1 = 0$ **and not (**$0 = s$ **or** $a - s = r - 1$**).** By Lemma 2.2 and Lemma 2.1 (d),

$$q^{2s+e-a}(1 + q^{-1})^2 - 1 \leqslant B \leqslant q^{2s+e-a} \left(\frac{q}{q-1}\right)^2 - 1.$$

Case ($s = d - r - 1$ **xor** $a - s - 1 = 0$**) and not (**$0 = s$ **or** $a - s = r - 1$**).** By Lemma 2.2, Lemma 2.1 (c), and Lemma 2.1 (d),

$$\begin{aligned} &q^{2s+e-a}(1 + q^{-1})^2 - (1 + 2q^{-1}) \\ &\leqslant B \leqslant q^{2s+e-a}(1 + 2q^{-1}) \cdot \frac{q}{q-1} - (1 + q^{-1}). \end{aligned}$$

Case ($0 = s$ xor $a - s = r - 1$) and
($s = d - r - 1$ xor $a - s - 1 = 0$).
By Lemma 2.2 and Lemma 2.1 (c),

$$q^{2s+e-a}(1+q^{-1}) - (1+2q^{-1}) \leqslant B \leqslant q^{2s+e-a}(1+2q^{-1}) - (1+q^{-1}).$$

Case not ($s = d - r - 1$ or $a - s - 1 = 0$ or $0 = s$ or $a - s = r - 1$). By Lemma 2.2, Lemma 2.1 (c),

$$q^{2s+e-a}(1+q^{-1})^2 - (1+2q^{-1})^2 \leqslant B \leqslant q^{2s+e-a}(1+2q^{-1})^2 - (1+q^{-1})^2.$$

With the given inequalities one can easily verify that the following two statements are true:

- If $2s + e - a \geqslant \frac{1}{2}$ and $q \geqslant 4$, then $B > 0$. Hence, $x > 0$.
- If $2s + e - a \leqslant -\frac{1}{2}$ and $q \geqslant 4$, then $B < 0$. Hence, $x < 0$.

For $q = 3$, the case $e \in \{\frac{1}{2}, \frac{3}{2}\}$ does not occur. Hence, $2s + e - a \geqslant \frac{1}{2}$ implies $2s + e - a \geqslant 1$ and $2s + e - a \leqslant -\frac{1}{2}$ implies $2s + e - a \leqslant -1$. If one uses the bound for $q = 3$ of Lemma 2.1 instead of the bound for $q \geqslant 4$ (i.e. replace all the factors $1 + 2q^{-1}$ by 2), then the proof for this case is the same. □

Corollary 5.33. *Let $q \geqslant 3$, $a < d$, and $r > 0$. Let $s_0 \in \{0, \ldots, a\}$ be the value for which $A(r, s_0, a)$ is largest. Then the following holds:*

$$|\lambda_r^a| \leqslant A(r, s_0, a).$$

Proof. This is a direct consequence of the formula for λ_r^a given in Theorem 5.25 and the unimodality of $A(r, s, a)$ given in Lemma 5.32. □

Proof of 5.28. The theorem was proven in Lemma 5.29 for $a = d - 1$, and in [66, p. 1295] for $a = 0$. Hence, we assume now that $1 \leqslant a \leqslant d - 2$. In view of Proposition 5.31 and since $\lambda_1^a < 0$, it suffices to show that $|\lambda_1^a| \geqslant |\lambda_r^a|$ for $2 \leqslant r \leqslant d - 2$. Corollary 5.33 gives

$$|\lambda_r^a| \leqslant \max\{A(r, s, a) : \max(a - r + 1, 0) \leqslant s \leqslant \min(a, d - r)\}.$$

Define

$$f(r,s) := \binom{d-r-s}{2} + (d-r-s)e + \binom{r+s-a}{2} + (d-r-s)s + (r+s-1-a)(a-s)$$

for all integers r, s with $a+1 \leqslant r+s \leqslant d$. We have $\lambda_1^a = -A(1,a,a)$ and Lemma 2.2 gives

$$A(1,a,a) \geqslant \frac{4}{3} q^{f(1,a)}.$$

Lemma 2.1 gives

$$A(r,s,a) \leqslant 4q^{f(r,s)}.$$

As $q \geqslant 3$, it suffices therefore to show that $f(1,a) \geqslant f(r,s)+1$ for all r, s with $2 \leqslant r \leqslant d-2$ and $a+1 \leqslant r+s \leqslant d$ and $0 \leqslant s \leqslant a$. Consider such a pair (r,s). An easy calculation gives

$$\begin{aligned} f(1,a) - f(r,s) &= (d-a-1)(r-1) - (r+s-a-1)(r-e-s) \\ &\geqslant (d-a-1)(r-1) - (r+s-a-1)(r-s). \end{aligned}$$

Denote the right hand side by $g(r,s)$. We can suppose $a < d-1$ by Lemma 5.29. If $s \geqslant 2$, then by $s \leqslant a$, $s+r \leqslant d$ and $r \geqslant 2$, we have that

$$\begin{aligned} g(r,s) &\geqslant (d-a-1)(r-1) - (r+s-a-1)(r-2) \\ &\geqslant (d-a-1)(r-1) - (d-a-1)(r-2) \\ &\geqslant d-a-1 \geqslant 1. \end{aligned}$$

If $s=1$, then by $d-1 > r > 1$

$$\begin{aligned} g(r,s) &= (d-a-1)(r-1) - (r-a)(r-1) \\ &= (r-1)(d-r-1) \geqslant r-1 \geqslant 1. \end{aligned}$$

If $s=0$, then by $a < d-1$

$$g(r,0) = (d-r-1)(r-1) + a \geqslant 1.$$

□

We have calculated the smallest eigenvalues and, therefore, Hoffman's bound, but it would be nice to have a simpler formula for the approximations of Section 5.7. This simplification will conclude this section.

Theorem 5.34. *Define* α *by* $\alpha \log(1+q^{-e-1}) \leqslant \log(1+q^{-e})$. *Set* $\gamma = 2$ *if* $q = 3$, *and* $\gamma = 1 + 2q^{-1}$ *otherwise. Let* $0 < t < d$.

(a) *We have*

$$c_{d,t} \leqslant -\lambda_{\min}(1+q^{-e})^{\frac{\alpha}{\alpha-1}},$$

where $\lambda_{\min} := \min_r \lambda_r^{d-t-1}$ *is the smallest eigenvalue of the matrix* $\sum_{s=t+1}^{d} A_s$.

(b) *Suppose* $q \geqslant 3$. *If* t *odd or* $e \geqslant 1$, *then*

$$c_{d,t} \leqslant \gamma q^{t(d-t-1)+\binom{t}{2}+te}(1+q^{-e})^{\frac{\alpha}{\alpha-1}}.$$

(c) *Suppose* $q \geqslant 3$. *If* t *even and* $e \leqslant 1$, *then*

$$c_{d,t} \leqslant \gamma q^{t(d-t-1)+\binom{t+1}{2}}(1+q^{-e})^{\frac{\alpha}{\alpha-1}}.$$

(d) *The same statements hold for* $q = 2$, $\gamma = \frac{111}{32}$, *and* $t = d-1$.

Proof. Here we have $a = d - t - 1$. Furthermore, by Theorem 5.25

$$k = \lambda_0^{d-t-1} \geqslant q^{\binom{d}{2}+de}.$$

By Proposition 5.27, Theorem 5.25, Lemma 2.7 and $\lambda_{\min} < 0$, we have

$$c_{d,t} \leqslant \frac{n\lambda_{\min}}{\lambda_{\min}-k} \leqslant -\lambda_{\min}\frac{n}{k} \leqslant -\lambda_{\min}(1+q^{-e})^{\frac{\alpha}{\alpha-1}}.$$

By Theorem 5.28 and Lemma 2.1, we have

$$\begin{aligned} -\lambda_{\min} &= \begin{bmatrix} d-1 \\ t \end{bmatrix} q^{\binom{t}{2}+te} \\ &\leqslant \gamma q^{t(d-t-1)+\binom{t}{2}+te} \end{aligned}$$

if t odd or $e \geqslant 1$, respectively,

$$\begin{aligned}-\lambda_{\min} &= (-1)^t \begin{bmatrix} d-1 \\ t \end{bmatrix} q^{\binom{t+1}{2}} \\ &\leqslant \gamma q^{t(d-t-1)+\binom{t+1}{2}}\end{aligned}$$

if t even and $e \leqslant 1$.

Part (d) can be proven in the same way by using the eigenvalues provided in Theorem 1.23 instead of the ones provided in Theorem 5.28, which is limited to $q \geqslant 3$. □

5.7 PROOF OF THEOREM 5.1

In this section we want to specify the q, d, and t for which our previous results are non-trivial statements. We shall do so by providing lower, respectively, upper bounds on all the parameters used in Theorem 5.12 and Theorem 5.24. For this we shall provide some upper estimates for $b_1^0, b_2^0, b_1^1, b_2^1, b_3^1$. Throughout this section, q is fixed and we define α and γ as follows.

(a) Let $\gamma = \frac{111}{32}$ if $q = 2$, $\gamma = 2$ if $q = 3$, and $\gamma = 1 + 2q^{-1}$ if $q \geqslant 4$.

(b) Let α be chosen as $\alpha \log(1 + q^{-e-1}) = \log(1 + q^{-e})$. We shall apply Theorem 5.34, where we only demand $\alpha \log(1 + q^{-e-1}) \leqslant \log(1 + q^{-e})$. This is more convenient for numerical approximations as we then do not have to calculate the exact values of the logarithms (which is usually not possible).

Lemma 5.35. *Let $5t \leqslant 2d + 1$. Then we have the following.*

(a) Suppose that t is even. Then we have

$$\psi^0 \leqslant q^{\frac{3}{4}t^2 + \frac{t}{2}(d-2t) - (d - \frac{5}{2}t + 2)} \frac{\gamma^2}{1 - q^{-2}}.$$

(b) Suppose that t is odd. Then we have

$$\psi^1 \leqslant q^{(\frac{t}{2} - \frac{1}{2})(\frac{3}{2}t - \frac{3}{2}) + (d-2t+1)(\frac{t}{2} - \frac{1}{2}) - (d - \frac{5}{2}t + \frac{7}{2})} \frac{\gamma^2}{q^2 - 1}.$$

Proof. Let t be even. For integers i with $1 \leqslant i \leqslant \frac{t}{2} - 1$, Lemma 2.1 shows that

$$\begin{aligned}
&q^{(\frac{t}{2}-1-i)(\frac{t}{2}-i)} \begin{bmatrix} d-2t+1 \\ t/2-i \end{bmatrix} \begin{bmatrix} t/2-1 \\ i \end{bmatrix} \\
&\leqslant q^{(\frac{t}{2}-1-i)(\frac{t}{2}-i)+(\frac{t}{2}-i)(d-\frac{5}{2}t+1+i)+i(\frac{t}{2}-1-i)}\gamma^2 \\
&= q^{\frac{t}{2}(d-2t)-i(d-\frac{5}{2}t+1+i)}\gamma^2.
\end{aligned}$$

Hence by Lemma 5.3 and $5t \leqslant 2d$, we have

$$\begin{aligned}
\psi^0 &= q^{\frac{3}{4}t^2} \sum_{i=1}^{t/2-1} q^{(\frac{t}{2}-1-i)(\frac{t}{2}-i)} \begin{bmatrix} d-2t+1 \\ t/2-i \end{bmatrix} \begin{bmatrix} t/2-1 \\ i \end{bmatrix} \\
&\leqslant q^{\frac{3}{4}t^2+\frac{t}{2}(d-2t)}\gamma^2 \sum_{i=1}^{t/2-1} q^{-i(d-\frac{5}{2}t+1+i)} \\
&\leqslant q^{\frac{3}{4}t^2+\frac{t}{2}(d-2t)-(d-\frac{5}{2}t+2)}\gamma^2 \sum_{i=0}^{t/2-2} q^{-2i} \\
&\leqslant q^{\frac{3}{4}t^2+\frac{t}{2}(d-2t)-(d-\frac{5}{2}t+2)} \frac{\gamma^2}{1-q^{-2}}.
\end{aligned}$$

This shows part (a).

Let t be odd. Similarly by Lemma 2.1, we have

$$\begin{aligned}
&q^{(\frac{t}{2}-\frac{3}{2}-i)(\frac{t}{2}-\frac{1}{2}-i)} \begin{bmatrix} d-2t+2 \\ (t-1)/2-i \end{bmatrix} \begin{bmatrix} t/2-\frac{3}{2} \\ i \end{bmatrix} \\
&\leqslant q^{(\frac{t}{2}-\frac{3}{2}-i)(\frac{t}{2}-\frac{1}{2}-i)+(\frac{t}{2}-\frac{1}{2}-i)(d-\frac{5}{2}t+\frac{5}{2}+i)+i(\frac{t}{2}-\frac{3}{2}-i)}\gamma^2 \\
&= q^{(d-2t+1)(\frac{t}{2}-\frac{1}{2})-i(d-\frac{5}{2}t+\frac{5}{2}+i)}\gamma^2.
\end{aligned}$$

Hence by Lemma 5.3 and Lemma 2.1, we have

$$\begin{aligned}
\psi^1 &= q^{(\frac{t}{2}-\frac{1}{2})(\frac{3}{2}t-\frac{3}{2})} \\
&\quad \cdot \sum_{i=1}^{t/2-3/2} q^{(\frac{t}{2}-\frac{3}{2}-i)(\frac{t}{2}-\frac{1}{2}-i)} \begin{bmatrix} d-2t+2 \\ (t-1)/2-i \end{bmatrix} \begin{bmatrix} \frac{t}{2}-\frac{3}{2} \\ i \end{bmatrix} \\
&\leqslant q^{(\frac{t}{2}-\frac{1}{2})(\frac{3}{2}t-\frac{3}{2})+(d-2t+1)(\frac{t}{2}-\frac{1}{2})} \gamma^2 \sum_{i=1}^{t/2-3/2} q^{-i(d-\frac{5}{2}t+\frac{5}{2}+i)} \\
&\leqslant q^{(\frac{t}{2}-\frac{1}{2})(\frac{3}{2}t-\frac{3}{2})+(d-2t+1)(\frac{t}{2}-\frac{1}{2})-(d-\frac{5}{2}t+\frac{7}{2})} \gamma^2 \sum_{i=0}^{t/2-5/2} q^{-2i} \\
&\leqslant q^{(\frac{t}{2}-\frac{1}{2})(\frac{3}{2}t-\frac{3}{2})+(d-2t+1)(\frac{t}{2}-\frac{1}{2})-(d-\frac{5}{2}t+\frac{7}{2})} \frac{\gamma^2}{1-q^{-2}}.
\end{aligned}$$

This shows part (b). □

Recall that the numbers b_i^0 are only defined for t even and that the numbers b_i^1 are only defined for t odd.

Lemma 5.36. *Let* $0 \leqslant 5t \leqslant 2d$.

(a) If $q \geqslant 3$, *then*

$$\begin{aligned}
b_1^0 &\leqslant q^{(\frac{t}{2}-1)(d-2t+1)+t(t-2)+\binom{t}{2}+te} \gamma^2 (1+q^{-e})^{\frac{\alpha}{\alpha-1}} \text{ if } e \geqslant 1, \\
b_1^0 &\leqslant q^{(\frac{t}{2}-1)(d-2t+1)+t(t-2)+\binom{t+1}{2}} \gamma^2 (1+q^{-e})^{\frac{\alpha}{\alpha-1}} \text{ if } e \leqslant 1, \\
b_1^1 &\leqslant q^{\frac{t-3}{2}(d-2t+2)+(t-3)t+\binom{t}{2}+te} \gamma^2 (1+q^{-e})^{\frac{\alpha}{\alpha-1}}, \\
b_2^0 &\leqslant q^{e\frac{t}{2}+\binom{t/2}{2}+\frac{2t(d+5)-t^2-4d-8}{4}} \frac{\gamma^2}{1-q^{-2}}, \\
b_2^1 &\leqslant q^{e\frac{t+1}{2}+\binom{(t+1)/2}{2}+\frac{2t(d+5)-t^2-6d-13}{4}} \frac{\gamma^2}{1-q^{-2}} (1+q^{-e})^{\frac{\alpha}{\alpha-1}}, \\
b_3^1 &\leqslant q^{e\frac{t-1}{2}+\binom{(t-1)/2}{2}+\frac{2t(d+1)-t^2-2d-1}{4}} \gamma.
\end{aligned}$$

(b) *If* $q = 2$, *then*

$$b_1^0 \leqslant q^{(\frac{t}{2}-1)(d-2t+1)+\binom{2t-2}{2}+(2t-2)e}\gamma^2(1+q^{-e})^{\frac{\alpha}{\alpha-1}} \text{ if } e \geqslant 1,$$
$$b_1^0 \leqslant q^{(\frac{t}{2}-1)(d-2t+1)+\binom{2t-1}{2}}\gamma^2(1+q^{-e})^{\frac{\alpha}{\alpha-1}} \text{ if } e \leqslant 1,$$
$$b_1^1 \leqslant q^{\frac{t-3}{2}(d-2t+2)+\binom{2t-3}{2}+(2t-3)e}\gamma^2(1+q^{-e})^{\frac{\alpha}{\alpha-1}},$$
$$b_2^0 \leqslant q^{e\frac{t}{2}+\binom{t/2}{2}+\frac{2t(d+5)-t^2-4d-8}{4}}\frac{\gamma^2}{1-q^{-2}},$$
$$b_2^1 \leqslant q^{e\frac{t+1}{2}+\binom{(t+1)/2}{2}+\frac{2t(d+5)-t^2-6d-13}{4}}\frac{\gamma^2}{1-q^{-2}}(1+q^{-e})^{\frac{\alpha}{\alpha-1}},$$
$$b_3^1 \leqslant q^{e\frac{t-1}{2}+\binom{(t-1)/2}{2}+\frac{2t(d+1)-t^2-2d-1}{4}}\gamma.$$

Proof. First consider the case $q > 2$. Apply Theorem 5.34, $c_{x,0} = 1$, Lemma 2.7, Lemma 2.1, and Lemma 5.35 on the following to obtain the stated upper bounds.

$$b_1^0 = \begin{bmatrix} d-\frac{3}{2}t \\ t/2-1 \end{bmatrix} c_{2t-1,t}$$
$$b_1^1 = \begin{bmatrix} d-\frac{3}{2}t+\frac{1}{2} \\ (t-3)/2 \end{bmatrix} c_{2t-2,t}$$
$$b_2^0 = \psi^0 q^{e\frac{t}{2}+\binom{t/2}{2}}$$
$$b_2^1 = \psi^1 \omega(d, t/2+\frac{1}{2}) = \psi^1 \prod_{i=0}^{t/2-\frac{1}{2}} (q^{i+e}+1)$$
$$\begin{aligned} b_3^1 &= \overline{\psi}^1 q^{e\frac{t-1}{2}+\binom{(t-1)/2}{2}} \\ &= q^{(\frac{3}{2}t-\frac{1}{2})(\frac{t}{2}-\frac{1}{2})} \begin{bmatrix} d-\frac{3}{2}t+\frac{1}{2} \\ (t-1)/2 \end{bmatrix} q^{e\frac{t-1}{2}+\binom{(t-1)/2}{2}} \end{aligned}$$

Since we can not apply Theorem 5.34 (a)–(c) for $q = 2$ for all t, we use $c_{2t-1,t} < c_{2t-1,2t-2}$, respectively, $c_{2t-2,t} < c_{2t-2,2t-3}$ together with Theorem 5.34 (d) and Theorem 1.23 for the approximation. By Theorem 1.23, the eigenvalues for the disjointness graph of a polar space of rank d and type e are

$$(-1)^r q^{\binom{d-r}{2}+\binom{r}{2}+e(d-r)}.$$

for $r \in \{0, 1, \ldots, d\}$. Hence, the smallest eigenvalue for the bound on $c_{2t-1,2t-2}$ is

$$\begin{aligned} &-q^{\binom{2t-2}{2}+e(2t-2)} && \text{for } t \text{ even and } e \geqslant 1, \\ &-q^{\binom{2t-1}{2}} && \text{for } t \text{ even and } e \leqslant 1. \end{aligned}$$

By Theorem 5.34 (d), the approximations for b_1^0 follow. Similarly, the smallest eigenvalue for the bound on $c_{2t-2,2t-3}$ is

$$-q^{\binom{2t-3}{2}+e(2t-3)}.$$

By Theorem 5.34 (d), the approximation for b_1^1 follows. □

Additionally, we need lower bounds for the size of our examples. Recall that the case $t = d - 1$ is not covered by Theorem 5.1 and that the case $t = 1$ is trivial, so we can assume $2 \leqslant t \leqslant d - 2$.

Lemma 5.37. *Let $q \geqslant 2$. Let $d - 2 \geqslant t \geqslant 2$. Example 5.11 has size at least*

$$y^0 := \begin{bmatrix} d \\ \frac{t}{2} \end{bmatrix} q^{e\frac{t}{2}+\binom{t/2}{2}} \geqslant q^{e\frac{t}{2}+\binom{t/2}{2}+\frac{t}{2}(d-\frac{t}{2})}(1+q^{-1}).$$

Example 5.19 has size at least

$$\begin{aligned} y^1 &:= \begin{bmatrix} d-1 \\ \frac{t-1}{2} \end{bmatrix} q^{e\frac{t+1}{2}+\binom{\frac{t+1}{2}}{2}}(1+q^{-e}) \\ &\geqslant q^{e\frac{t+1}{2}+\binom{\frac{t+1}{2}}{2}+\frac{t-1}{2}(d-\frac{t+1}{2})}(1+q^{-1})(1+q^{-e}). \end{aligned}$$

If $q = 2$, then

$$y^1 \geqslant q^{e\frac{t+1}{2}+\binom{\frac{t+1}{2}}{2}+\frac{t-1}{2}(d-\frac{t+1}{2})}(2-\frac{1}{q^2})(1+q^{-e}).$$

Proof. **Case t even.** Let Y' be the set of all generators which meet a given generator in exactly dimension $d - \frac{t}{2}$. Obviously, Y' has less elements than Example 5.11. We shall show $|Y'| \geqslant y^0$. By Lemma 1.24, there are exactly

$$\begin{bmatrix} d \\ \frac{t}{2} \end{bmatrix} q^{e\frac{t}{2}+\binom{t/2}{2}}$$

such generators. Lemma 2.2 shows the remaining inequality for y^0.

Case t **odd.** Let U be a $(d-1)$-dimensional totally isotropic subspace. Let G be a generator with $U \subseteq G$. Let Y' be the set of all generators which meet U in dimension $d - \frac{t}{2} - \frac{1}{2}$. Obviously, Y' has at most as many elements as Example 5.19. We shall show $|Y'| \geqslant y^1$.

We have $\left[\begin{smallmatrix} d-1 \\ \frac{t-1}{2} \end{smallmatrix}\right]$ possibilities to choose a $(d - \frac{t}{2} - \frac{1}{2})$-dimensional subspace T of U. Let $H \in Y'$ with $U \cap H = T$. Then $G \cap H = T$ or $V := G \cap H$ has dimension $\dim(T) + 1$ and satisfies $V \cap U = T$. In the quotient geometry on T, Corollary 1.24 shows that there exist

$$q^{e\frac{t+1}{2} + \binom{\frac{t+1}{2}}{2}}$$

generators H with $H \cap G = T$. The number of subspaces V of G with $\dim(V) = \dim(T) + 1$ and $V \cap U = T$ is

$$[\frac{t+1}{2}] - [\frac{t-1}{2}] = q^{\frac{t-1}{2}}$$

as can be seen in the quotient geometry on T. For each such V, Corollary 1.24 applied to the quotient geometry of V shows that there are

$$q^{e\frac{t-1}{2} + \binom{\frac{t-1}{2}}{2}}$$

generators H with $H \cap G = V$. Hence, we find that Y' has at least

$$\begin{aligned} &\begin{bmatrix} d-1 \\ \frac{t-1}{2} \end{bmatrix} \left(q^{e\frac{t+1}{2} + \binom{\frac{t+1}{2}}{2}} + q^{\frac{t-1}{2}} \cdot q^{e\frac{t-1}{2} + \binom{\frac{t-1}{2}}{2}} \right) \\ = &\begin{bmatrix} d-1 \\ \frac{t-1}{2} \end{bmatrix} q^{e\frac{t+1}{2} + \binom{\frac{t+1}{2}}{2}} \left(1 + q^{-e}\right) \end{aligned}$$

elements. Lemma 2.2 shows the remaining inequality. In particular, if $q = 2$, then the condition $d - 1 > t \geqslant 3$ implies $d - \frac{t}{2} - \frac{1}{2} \geqslant 2$. Hence, by Lemma 2.2,

$$\begin{bmatrix} d-1 \\ \frac{t-1}{2} \end{bmatrix} \geqslant (2 - q^{-2}) q^{(d - \frac{t}{2} - \frac{1}{2})(\frac{t}{2} - \frac{1}{2})}.$$

$\square$

All left to do is to compare $b_1^0 + b_2^0$, respectively, $2b_1^1 + b_2^1 + b_3^1$ to the sizes of the examples (y^0, respectively, y^1) using the given upper, respectively, lower bounds. Then Theorem 5.12 and Theorem 5.24 yield our last theorem. Hence, we compare all degrees of the bounds in q to y^0, respectively, y^1. This yields for $q \geqslant 3$,

$$\begin{aligned}
\delta_1^0 &:= \deg(y^0) - \deg(b_1^0) = \begin{cases} d + 1 - \frac{2e+1}{4}t - \frac{5}{8}t^2 \text{ if } e \geqslant 1, \\ d + 1 - \frac{5-2e}{4}t - \frac{5}{8}t^2 \text{ if } e \leqslant 1, \end{cases} \\
\delta_2^0 &:= \deg(y^0) - \deg(b_2^0) = d + 2 - \frac{5}{2}t, \\
\delta_1^1 &:= \deg(y^1) - \deg(b_1^1) = d + \frac{25}{8} + e/2 - \frac{(e+1)t}{2} - \frac{5}{8}t^2, \\
\delta_2^1 &:= \deg(y^1) - \deg(b_2^1) = d + \frac{7}{2} - \frac{5}{2}t, \\
\delta_3^1 &:= \deg(y^1) - \deg(b_3^1) = e.
\end{aligned}$$

For $q = 2$ only the following values are different from the values for $q \geqslant 3$:

$$\begin{aligned}
\delta_1^0 &:= \deg(y^0 - b_1^0) = \begin{cases} d - 2 + 2e - \frac{6e-9}{4}t - \frac{9}{8}t^2 \text{ if } e \geqslant 1, \\ d + \frac{2e+1}{4}t - \frac{9}{8}t^2 \text{ if } e \leqslant 1, \end{cases} \\
\delta_1^1 &:= \deg(y^1 - b_1^1) = d + \frac{7}{2}e - \frac{23}{8} + \frac{6-3e}{2}t - \frac{9}{8}t^2.
\end{aligned}$$

These approximations make it clear that Theorem 5.12 and Theorem 5.24 are non-trivial for d large and t fixed. In the following we want to be more specific about the necessary size of d. Recall $\gamma = \frac{111}{32}$ if $q = 2$, $\gamma = 2$ if $q = 3$, and $\gamma = 1 + 2q^{-1}$ if $q \geqslant 4$.

Lemma 5.38. *Let z be a integer. Let $q \geqslant 2$.*

(a) The equation

$$q^z(1 + q^{-1}) \geqslant \gamma^2((1 + q^{-e})^{\frac{\alpha}{\alpha-1}} + \frac{1}{1-q^{-2}})$$

is satisfied if $z \geqslant 7$ or if $z \geqslant 3$ and $q \geqslant 3$.

(b) *The equation*

$$q^z(1+q^{-1})(1+q^{-e}) \geqslant \gamma q^{z-e} + \gamma^2(1+q^{-e})^{\frac{\alpha}{\alpha-1}}(2+\tfrac{1}{1-q^{-2}})$$

is satisfied if $z \geqslant 4$ *and* $q \geqslant 3$.

(c) *The equation*

$$q^z(2-q^{-2})(1+q^{-e}) \geqslant \gamma q^{z-e} + \gamma^2(1+q^{-e})^{\frac{\alpha}{\alpha-1}}(2+\tfrac{1}{1-q^{-2}})$$

is satisfied if $z \geqslant 13$.

Proof. One can check that the difference between the left side and the right side of the inequality is monotonically increasing for $z \geqslant 4$ in q and e, so one only has to check $q \in \{2,3,4\}$ and $e = 0$. We shall show the monotonicity in the following.

Step 1: We can assume $e = 0$. In the equation in (a) only the term

$$(1+q^{-e})^{\frac{\alpha}{\alpha-1}} \tag{5.1}$$

depends on e. By Lemma 2.6, the term (5.1) decreases if we increase e. Hence, increasing e decreases the right side of the equation in (a), but leaves the left side constant.

For the same statement for the equality in (b) we show that the terms

$$q^z(1+q^{-1}) - \gamma^2(1+q^{-e})^{\frac{\alpha}{\alpha-1}}(2+\tfrac{1}{1-q^{-2}}) \text{ and}$$
$$q^{z-e}(1+q^{-1}) - \gamma q^{z-e}$$

both increase by increasing e. The first term, again by Lemma 2.6, is monotonically increasing in e. For the second term, $\gamma > 1 + q^{-1}$ shows the claim.

The monotonicity in e for (c) can be seen as in (b), using $\gamma = \frac{111}{32} > \frac{7}{4} = 1 + 2q^{-1}$ for $q = 2$.

Step 2: We can assume $q \in \{2,3,4\}$. We shall show that the difference between the left side and the right side increases if we substitute q by $q+1$ for $q \geqslant 4$. By Step 1 we can assume $e = 0$.

The right side of the equation in (a) is monotonically decreasing in q by Lemma 2.6. The right side is increasing as seen by

$$\begin{aligned}&(q+1)^z(1+(q+1)^{-1}) - q^z(1+q^{-1})\\ &= (q+1)^{z-1}(q+2) - q^{z-1}(q+1) > 0.\end{aligned}$$

This shows the assertion.

For (b) and (c) we only show that

$$2q^z(1+q^{-1}) - \gamma q^z$$

increases by increasing q. This is sufficient as the remaining term on the right side of equations decreases by increasing q by Lemma 2.6. By $2(1+q^{-1}) = 2+2q^{-1} > 1+2q^{-1} = \gamma$ for $q \geqslant 4$. This shows the last part of the assertion.

All that remains is to show the inequalities for $e = 0$ and $q \in \{2, 3, 4\}$. The author did this by computer. To check the given inequalities numerically, one has to pay some attention to numerical stability. Here it is important to keep in mind that all of the used results only require $\alpha \log(1+q^{-e-1}) \leqslant \log(1+q^{-e})$, which make numerical approximations possible. $\square$

Proof of Theorem 5.1. **Case** $t = 1$**:** By Lemma 5.5, the largest $(d, 1)$-EKR set is the set of all generators through a fixed $(d-1)$-space.

Case $t \geqslant 2$**:** We shall apply Theorem 5.12, respectively, Theorem 5.24.

Let z be as in Lemma 5.38. If t is even, then according to the appropriate part of Lemma 5.38, Lemma 5.37, and Lemma 5.36

$$\min(\delta_1^0, \delta_2^0) \geqslant z$$

is a sufficient condition for $b_1^0 + b_2^0 < |Y|$. If z is odd, then

$$\min(\delta_1^1, \delta_2^1, \delta_3^1) \geqslant z$$

is a sufficient condition for $2b_1^1 + b_2^1 + b_3^1 < |Y|$.

The given conditions on t imply the above inequalities for $t \geqslant 2$ as we shall see in the following.

If t is even and $q = 2$, then z has to be at least 7 by Lemma 5.38 (a). The given condition is $2 \leqslant t \leqslant \sqrt{\frac{8d}{9}} - 2$. Since we may assume $t \geqslant 2$, this implies $d \geqslant \frac{3^2 \cdot 9}{8}$. A simple calculation shows that we have $\min(\delta_1^0, \delta_2^0) \geqslant 7$ under these conditions.

If t is even and $q \geqslant 3$, then z has to be at least 3 by Lemma 5.38 (a). The given condition is $2 \leqslant t \leqslant \sqrt{\frac{8d}{5}} - 2$. Since we may assume

$t \geqslant 2$, this implies $d \geqslant \frac{4^2 \cdot 5}{8}$. A simple calculation shows that we have $\min(\delta_1^0, \delta_2^0) \geqslant 4$ under the given conditions.

If t is odd and $q = 2$, then z has to be at least 13 by Lemma 5.38 (c). The given condition is $3 \leqslant t \leqslant \sqrt{\frac{8d}{9}} - 2$. Since we may assume $t \geqslant 3$, this implies $d \geqslant \frac{5^2 \cdot 9}{8}$. A simple calculation shows that we have $\min(\delta_1^1, \delta_2^1) \geqslant 13$ under these conditions.

If t is even and $q \geqslant 3$, then z has to be at least 4 by Lemma 5.38 (b). The given condition is $3 \leqslant t \leqslant \sqrt{\frac{8d}{5}} - 2$. Since we may assume $t \geqslant 3$, this implies $d \geqslant \frac{5^3}{8}$. A simple calculation shows that we have $\min(\delta_1^1, \delta_2^1) \geqslant 11$ under the given conditions. □

Remark 5.39. *(a) Obviously, even the trivial upper bound for $c_{2t-1,t}$, i.e. the number of generators in a polar space of rank $2t-1$, is independent of d. If one uses this bound instead of Theorem 5.34 to bound $c_{2t-1,t}$, then the restriction on t is approximately $t \leqslant \sqrt{\frac{8d}{9}}$ as in the $q = 2$ case.*

(b) If one uses the linear programming bound instead of Hoffman's bound to approximate $c_{d,t}$, then computer results suggest that the conditions on t in Theorem 5.1 should simplify to approximately $t \leqslant 2\sqrt{2d}$ for d large.

(c) If one could prove that $c_{2t-1,t}$ is the size of Example 5.11, respectively, that $c_{2t-1,t}$ is the size of Example 5.19, then the conditions on t would improve to approximately $t \leqslant \frac{2}{5}d$. So it would be sufficient to focus on these cases to improve the results significantly.

5.8 CONCLUDING REMARKS

This project was started with the hope that it would be simple to generalize the classification of $(d, d-1)$-EKR sets of maximum size provided in [66] by applying Hoffman's bound or one of its generalizations since Hoffman's bound is tight in this case [68] if $e \neq \frac{1}{2}$. It turns out that for nearly all (d, t)-EKR sets Hoffman's bound is far larger than the largest known examples.

Linear programming can be used to obtain better algebraic bounds for all d. While computer results suggest that these upper bounds should be able to improve Theorem 5.1 to approximately $t \leqslant 2\sqrt{2d}$, even these bounds are still far away from the largest known examples. Hence, one has to rely explicitly on the geometrical properties of polar spaces for the classification. It might be very interesting to find a purely algebraical proof of the presented results, since our approach stops working as soon as t is too large compared to d, while techniques from algebraic combinatorics seem to work the best when t is large compared to d.

In general, a classification of all (d, t)-EKR sets seems to be very desirable, since the author conjectures that it would turn out to be the following, nice looking result.

Conjecture 5.40. *Let Y be a (d, t)-EKR set of maximum size. Then one of the following cases occurs:*

(a) We have that Y is a dictatorship.

(b) We have that t is even and Y is a d-junta.

(c) We have that t is odd and Y is a $(d-1)$-junta.

(d) We have that $e = 0$, $t = d-1$, d is odd and Y is the largest example for $Q^+(2d-1, q)$ as given in [66], i.e. the set of all Latin generators or the set of all Greek generators.

6 CROSS-INTERSECTING EKR SETS OF POLAR SPACES

A general overview over the topic of cross-intersecting EKR sets was given in Chapter 3. Recall that a *cross-intersecting EKR set of generators* is a pair (Y, Z) of sets of generators such that all $y \in Y$ and all $z \in Z$ intersect in at least a point. In this setting this chapter is only concerned with an upper bound on $|Y| \cdot |Z|$ and a classification of all cross-intersecting EKR sets reaching this bound.

An additional motivation for this problem is the following: as mentioned before the problem of EKR sets of maximum size on $H(2d-1, q^2)$ is still open for $d > 3$ odd. Let P be a point of $H(2d-1, q^2)$ and let X be an EKR set of $H(2d-1, q^2)$. Furthermore, let Y be the set of generators of X on P and Z the set of generators of X not on P. Now in the quotient geometry of P isomorphic to $H(2d-3, q^2)$ the projection of the generators of Y and Z onto the quotient geometry is a cross-intersecting EKR set. Hence both problems are related.

One last thing to point out is that the following does not provide tight upper bounds for cross-intersecting EKR sets in $H(2d-1, q^2)$ for all $d > 1$. The problem is very similar to the open problem of the maximum size of EKR sets in $H(9, q^2)$. Therefore, it could be reasonable to first solve the problem of the maximum size of cross-intersecting EKR sets in $H(7, q^2)$ and then generalize the technique to EKR sets in $H(9, q^2)$.

6.1 FIRST OBSERVATIONS

In this section we shall calculate tight upper bounds for all polar spaces except $H(2d-1, q^2)$, and classify all examples in case of equality. For all polar spaces except $H(2d-1, q^2)$ we can imitate the approach of Pepe, Storme, and Vanhove [66]. Recall from Section 1.3 that we have a natural ordering of the eigenspaces $V_0 (= \langle \mathbf{j} \rangle), V_1, \ldots, V_d$ of the association scheme which we defined on generators of a polar space of rank d.

Lemma 6.1. *Let $\mathcal{P}$ be a polar space over $\mathbb{F}_q$ with parameter e and let A_d be the adjacency matrix matrix of the disjointness graph of generators of $\mathcal{P}$. Then $k = q^{\binom{d}{2}+de}$ and we have the following:*

- *If* $\mathcal{P} = Q^+(2d-1,q)$*, then* $\lambda_b = q^{\binom{d}{2}}$*. Moreover, if* d *is even, then* $\lambda_b = \lambda_+ > -\lambda_-$ *and* $V_+ = V_d$*; if* d *is odd, then* $\lambda_b = \lambda_- > -\lambda_+$ *and* $V_- = V_d$*.*

- *If* $\mathcal{P} \in \{Q(2d,q), W(2d-1,q)\}$*, then* $\lambda_b = q^{\binom{d}{2}}$*. Moreover, if* d *is even, then* $\lambda_b = -\lambda_- = \lambda_+$*,* $V_- = V_1$ *and* $V_+ = V_d$*; if* d *is odd, then* $\lambda_b = -\lambda_- > \lambda_+$ *and* $V_- = V_0 \perp V_1 \perp V_d$*.*

- *If* $\mathcal{P} \in \{H(2d,q), Q^-(2d+1,q)\}$*, then* $\lambda_b = q^{\binom{d-1}{2}+de}$*. Moreover,* $\lambda_b = -\lambda_- > \lambda_+$ *and* $V_- = V_1$*.*

- *If* $\mathcal{P} = H(2d-1,q)$*, then* $\lambda_b = q^{\binom{d}{2}}$*. Moreover, if* d *is even, then* $\lambda_b = \lambda_+ > -\lambda_-$ *and* $V_+ = V_d$*; if* d *is odd, then* $\lambda_b = -\lambda_- > \lambda_+$ *and* $V_- = V_1 \perp V_d$*.*

Proof. The eigenvalues of A_d were given in Theorem 1.23 as

$$(-1)^r q^{\binom{d-r}{2}+\binom{r}{2}+e(d-r)}.$$

For $r = 0$ this is the eigenvalue that belongs to the all-one vector $\mathbf{j}$, so define k by

$$k = q^{\binom{d}{2}+de}.$$

For $e = 0$ note that the absolute eigenvalues for $r = 0$ and $r = d$ are equal. Therefore, the eigenspace belonging to k has dimension at least 2 which make k also the second largest absolute eigenvalue. Hence, we have the following for the different polar spaces. For $e = 0$ (i.e. $\mathcal{P} = Q^+(2d-1,q)$) the second largest absolute eigenvalue occurs if and only if $r = d$, for $e = 1$ (i.e. $\mathcal{P} \in \{Q(2d,q), W(2d-1,q)\}$) the second largest absolute eigenvalue occurs if and only if $r \in \{1,d\}$, for $e \in \{3/2, 2\}$ (i.e. $\mathcal{P} \in \{H(2d,q), Q^-(2d+1,q)\}$) the second largest absolute eigenvalue occurs if and only if $r = 1$. □

Using Proposition 1.12 and the classification of EKR sets of generators of maximum size given in [66] we get the following result.

Corollary 6.2. *Let* (Y, Z) *be a cross-intersecting EKR set of maximum size of a finite classical polar space* $\mathcal{P}$ *not isomorphic to* $Q(2d,q)$ *with* d *even,*

$W(2d-1,q)$ *with* d *even,* $Q^+(2d-1,q)$ *with* d *even, or* $H(2d-1,q^2)$, *where* $|Y| \cdot |Z|$ *reaches the bound in Theorem 6.3. Then* $Y = Z$, *and* Y *is an EKR set of maximum size.*

Proof. By Proposition 1.12 and Lemma 6.1, all cross-intersecting EKR, which reach the bound, are EKR sets of maximum size. These EKR sets exist as shown in [66]. □

The cases, which are summarized in the following result, remain.

Theorem 6.3. *Let* (Y, Z) *be a cross-intersecting EKR set of generators of maximum size of a polar space* $\mathcal{P}$, *with* $\mathcal{P}$ *isomorphic to* $Q(2d, q)$, d *even,* $W(2d-1, q)$, d *even, or* $Q^+(2d-1, q)$, d *even. Let* n *be the number of generators of* $\mathcal{P}$. *Then we have the following:*

- *If* $\mathcal{P} = Q^+(2d-1, q)$, *then* $\sqrt{|Y| \cdot |Z|}$ *is at most* $n/2$, *and if this bound is reached, then there are* $v_- \in V_1$ *and* $v_+ \in V_d$ *such that* $\chi_Y = \alpha\mathbf{j} + v_- + v_+$ *and* $\chi_Z = \mathbf{j}/2 + v_- - v_+$.
- *If* $\mathcal{P} \in \{Q(2d, q), W(2d-1, q)\}$, *then* $\sqrt{|Y| \cdot |Z|}$ *is at most the number of generators on a fixed point, and if this bound is reached, then there are* $v_- \in V_1$ *and* $v_+ \in V_d$ *such that* $\chi_Y = \alpha\mathbf{j} + v_- + v_+$ *and* $\chi_Z = \alpha\mathbf{j} + v_- - v_+$, *with* $\alpha = \frac{1}{q^d+1}$.

Proof. Apply Lemma 1.12 and Lemma 6.1. □

6.2 THE NON-EKR CASES

We shall continue to classify the more complicated cases, i.e. the cases where it is not already clear that the largest cross-intersecting EKR sets are EKR sets.

6.2.1 *The Hyperbolic Quadric of Even Rank*

As mentioned in Chapter 1 the generators of $Q^+(2d-1, q)$ can be partitioned into Latin generators X_1 and Greek generators X_2 with $|X_1| = |X_2| = n/2$. Recall that for $x_1 \in X_1$ and $x_2 \in X_2$ the codimension of the intersection of $x_1 \cap x_2$ is odd. Recall that for $x_1, x_2 \in X_1$ the codimension of the intersection of $x \cap y$ is even. This implies for d

even that (X_1, X_2) is a cross-intersecting EKR set of maximum size according to Theorem 6.3. There exist $x_1, x_2 \in X_1$ with $\dim(x_1 \cap x_2) = 0$ if d is even, so (X_1, X_1) is not a cross-intersecting EKR set.

Theorem 6.4. *Let (Y, Z) be a cross-intersecting EKR set of maximum size of $Q^+(2d-1, q)$ with d even. Then $Y = X_i$ and $Z = X_j$ for $\{i, j\} = \{1, 2\}$.*

Proof. By Theorem 6.3, we have $\chi_Y = \mathbf{j}/2 + \nu_- + \nu_+$ and $\chi_Z = \mathbf{j}/2 + \nu_- - \nu_+$. As in Theorem 16 of [66] V_0 is spanned by $\chi_{X_1} + \chi_{X_2}$, and V_d is spanned by $\chi_{X_1} - \chi_{X_2}$. Hence $\chi_Y, \chi_Z \in \{\chi_{X_1}, \chi_{X_2}\}$ as $\chi_Y, \chi_Z, \chi_{X_1}, \chi_{X_2}$ are 0-1-vectors with $\chi_{X_1} + \chi_{X_2} = \mathbf{j}$. Hence without loss of generality $Y = X_1$. Since (X_1, X_1) is not a cross-intersecting EKR set, we have $Z = X_2$. □

6.2.2 *The Parabolic Quadric and the Symplectic Polar Space of Even Rank*

Before we classify cross-intersecting EKR sets of generators of maximum size in parabolic quadrics and symplectic polar spaces, we mention a few simple lemmas. We include their proofs to make this thesis more self-contained.

Lemma 6.5. *Let $\chi \in \langle \mathbf{j} \rangle \perp V$ for some eigenspace V of an (extended weight) adjacency matrix A of a k-regular graph with n vertices associated with eigenvalue λ. Then the characteristic vector e_i of the i-th vertex satisfies*

$$e_i^T A \chi = \frac{\chi^T \mathbf{j}}{n}(k - \lambda) + \lambda e_i^T \chi.$$

Proof. By definition, $\chi = \frac{\chi^T \mathbf{j}}{n} \mathbf{j} + \nu$ for some $\nu \in V$. Then

$$\begin{aligned} e_i^T A \chi &= e_i^T A \left(\frac{\chi^T \mathbf{j}}{n} \mathbf{j} + \nu\right) \\ &= e_i^T \left(\frac{\chi^T \mathbf{j}}{n} k \mathbf{j} + \lambda \nu\right) \\ &= e_i^T \left(\frac{\chi^T \mathbf{j}}{n}(k - \lambda) \mathbf{j} + \lambda \chi\right) \\ &= \frac{\chi^T \mathbf{j}}{n}(k - \lambda) + \lambda e_i^T \chi. \end{aligned}$$ □

Recall for this section that all matrices A_i of a given association scheme have the same eigenspaces V_j.

Corollary 6.6. *Let $\chi, \psi \in \langle \mathbf{j} \rangle \perp V_- \perp V_+$ for some eigenspaces V_- and V_+ of a (weighted) adjacency matrix A of a k-regular graph with n vertices, where λ_- is the eigenvalue associated with V_- and λ_+ is the eigenvalue associated with V_+. If $\chi = \alpha\mathbf{j} + \nu_- + \nu_+$ and $\psi = \alpha\mathbf{j} + \nu_- - \nu_+$ for some $\alpha \in \mathbb{R}$, $\nu^- \in V_-$, and $\nu^+ \in V^+$, then*

$$e_i^T A\chi = \frac{(\chi+\psi)^T\mathbf{j}}{2n}(k-\lambda_-) + \frac{(\lambda_- + \lambda_+)}{2} e_i^T\chi + \frac{(\lambda_- - \lambda_+)}{2} e_i^T\psi,$$
$$e_i^T A\psi = \frac{(\chi+\psi)^T\mathbf{j}}{2n}(k-\lambda_-) + \frac{(\lambda_- + \lambda_+)}{2} e_i^T\psi + \frac{(\lambda_- - \lambda_+)}{2} e_i^T\chi.$$

Proof. We have $\chi + \psi \in \langle \mathbf{j} \rangle \perp V_-$ and $\chi - \psi \in V_+$. By Lemma 6.5 and $\mathbf{j}^T\nu^- = 0 = \mathbf{j}^T\nu^+$,

$$e_i^T A(\chi+\psi) = \frac{(\chi+\psi)^T\mathbf{j}}{n}(k-\lambda_-) + \lambda_- e_i^T(\chi+\psi)$$
$$e_i^T A(\chi-\psi) = \lambda_+ e_i^T(\chi-\psi).$$

Now the equations $2e_i^T A\chi = e_i^T A(\chi+\psi) + e_i^T A(\chi-\psi)$, $2e_i^T A\psi = e_i^T A(\chi+\psi) - e_i^T A(\chi-\psi)$ yield the assertion. □

Lemma 6.7. *Consider the adjacency matrices $\{A_0, A_1, \dots, A_d\}$ of $Q(2d, q)$ (or $W(2d-1, q)$). For the adjacency matrix A_{d-s} the eigenspace V_1 is associated with eigenvalue*

$$\lambda_{-,s} := -\begin{bmatrix} d-1 \\ s \end{bmatrix} q^{\binom{d-s}{2}} + \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{\binom{d-s+1}{2}},$$

the eigenspace V_d is associated with eigenvalue

$$\lambda_{+,s} := (-1)^{d-s} \begin{bmatrix} d \\ s \end{bmatrix} q^{\binom{d-s}{2}},$$

and the eigenspace $E_0 = \langle \mathbf{j} \rangle$ is associated with eigenvalue

$$k_s := \begin{bmatrix} d \\ s \end{bmatrix} q^{\binom{d-s+1}{2}} = \left(\begin{bmatrix} d-1 \\ s \end{bmatrix} + \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{d-s} \right) q^{\binom{d-s+1}{2}}.$$

Proof. By Theorem 1.23, the eigenvalue of V_j for A_i is

$$\sum_{u=\max(j-i,0)}^{\min(d-i,j)} (-1)^{j+u} \begin{bmatrix} d-j \\ d-i-u \end{bmatrix} \begin{bmatrix} j \\ u \end{bmatrix} \cdot q^{(u+i-j)(u+i-j+1)/2 + \binom{j-u}{2}}.$$

For $j = 0$, $j = 1$, $j = d$, and $i = d - s$, this formula yields the assertion. The last equality is an application of (1.16). □

If a cross-intersecting EKR set (Y, Z) of $Q(2d, q)$ satisfies $\chi_Y, \chi_Z \in V_0 \perp V_1$, then $Y = Z$, and Y is an EKR set of maximum size (as it satisfies the Hoffman bound 1.9 with equality). So only the case $\chi_Y, \chi_Z \in V_0 \perp V_1 \perp V_d$ remains. In the following denote V_1 by V_- and V_d by V_+. Furthermore, as in Lemma 1.12 we write

$$\begin{aligned}\chi_Y &= \alpha\mathbf{j} + v_- + v_+ \\ &= \frac{|Y|}{n}\mathbf{j} + v_- + v_+ \\ &= \frac{\lambda_b}{k + \lambda_b}\mathbf{j} + v_- + v_+\end{aligned}$$

and

$$\begin{aligned}\chi_Z &= \alpha\mathbf{j} + v_- - v_+ \\ &= \frac{|Z|}{n}\mathbf{j} + v_- - v_+ \\ &= \frac{\lambda_b}{k + \lambda_b}\mathbf{j} + v_- - v_+\end{aligned}$$

with $v_- \in V_-$ and $v_+ \in V_+$.

Proposition 6.8. *Let (Y, Z) be a cross-intersecting EKR set of $Q(2d, q)$ or $W(2d-1, q)$, d even, of maximum size such that $Y \cap Z \neq Y$. Let $G \in Y \setminus Z$.*

(a) If $d - s$ is even, then G meets 0 elements of Z in dimension s.

(b) If $d - s$ is odd, then G meets 0 elements of Y in dimension s.

(c) If $d - s$ is even, then G meets

$$\begin{bmatrix} d \\ s \end{bmatrix} q^{\binom{d-s}{2}}$$

elements of Y in dimension s.

(d) If $d-s$ is odd, then G meets

$$\begin{bmatrix} d \\ s \end{bmatrix} q^{\binom{d-s}{2}}$$

elements of Z in dimension s.

In particular, $Y \cap Z = \emptyset$.

Proof. We can calculate these numbers with Lemma 6.7 and Corollary 6.6 by choosing $\chi_{\{G\}}$ as e_i. For A_{d-s} the parameters are given by

$$\begin{aligned}
k_s - \lambda_{-,s} &= \left(\begin{bmatrix} d-1 \\ s \end{bmatrix} + \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{d-s} \right) q^{\binom{d-s+1}{2}} \\
&\quad + \begin{bmatrix} d-1 \\ s \end{bmatrix} q^{\binom{d-s}{2}} - \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{\binom{d-s+1}{2}} \\
&= q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} (q^{d-s}+1) \\
&\quad + q^{\binom{d-s+1}{2}} \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} (q^{d-s}-1) \\
&\overset{\text{Def.}}{=} q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} (q^{d-s}+1) \\
&\quad + q^{d-s} \cdot q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} (q^{s}-1) \\
&= q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} (q^{d}+1),
\end{aligned}$$

for $d-s$ even

$$\begin{aligned}
\lambda_{-,s} + \lambda_{+,s} &\overset{(1.16)}{=} 2 \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{\binom{d-s+1}{2}}, \\
\lambda_{-,s} - \lambda_{+,s} &\overset{(1.16)}{=} -2 \begin{bmatrix} d-1 \\ s \end{bmatrix} q^{\binom{d-s}{2}},
\end{aligned}$$

for $d-s$ odd

$$\begin{aligned}
\lambda_{-,s} + \lambda_{+,s} &\overset{(1.16)}{=} -2 \begin{bmatrix} d-1 \\ s \end{bmatrix} q^{\binom{d-s}{2}}, \\
\lambda_{-,s} - \lambda_{+,s} &\overset{(1.16)}{=} 2 \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{\binom{d-s+1}{2}}.
\end{aligned}$$

Recall that

$$\frac{(\chi_Y + \chi_Z)^T \mathbf{j}}{2n} = \frac{\lambda_+}{k + \lambda_+} = \frac{q^{\binom{d}{2}}}{q^{\binom{d+1}{2}} + q^{\binom{d}{2}}} = \frac{1}{q^d + 1}.$$

Hence by Corollary 6.6 and $g \notin Z$,

$$\begin{aligned} e_i^T A_{d-s} \chi_Y &= \frac{k_s - \lambda_{-,s}}{q^d + 1} + \frac{(\lambda_- + \lambda_+)}{2} e_i^T \chi \\ &= q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} + \frac{(\lambda_- + \lambda_+)}{2} e_i^T \chi. \end{aligned}$$

If $d - s$ is even and $e_i^T \chi = 1$, then by (1.16)

$$e_i^T A_{d-s} \chi_Y = q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} + q^{\binom{d-s+1}{2}} \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} = q^{\binom{d-s}{2}} \begin{bmatrix} d \\ s \end{bmatrix}.$$

If $d - s$ is odd and $e_i^T \chi = 1$, then

$$e_i^T A_{d-s} \chi_Y = q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} - q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} = 0.$$

All remaining cases follow from the fact that $|Y| \cdot |Z|$ has maximum size and symmetry. No element of Y meets g in the same dimension as an element of Z, hence $Y \cap Z = \emptyset$. □

Now we have obtained a strong combinatorial information about cross-intersecting EKR sets (Y, Z) of maximum size which are not EKR sets. By adding some geometrical arguments this leads to a complete classification of cross-intersecting EKR sets in these parabolic and symplectic polar spaces.

Lemma 6.9. *Let (Y, Z) be a largest cross-intersecting EKR set of $Q(2d, q)$ or $W(2d-1, q)$, d even. Let $G, H \in Y$ disjoint (see Proposition 6.8 (c)). Let $\pi_1, \ldots, \pi_{[d]} \subseteq G$ be the $[d]$ subspaces of dimension $d-1$ of G. Then the following holds:*

(a) Exactly $[d]$ elements $z_1, \ldots, z_{[d]}$ of Z meet G in dimension $d-1$.

(b) We have $\{z_i : i \in \{1, \ldots, [d]\}\} = \{\langle \pi_i, \pi_i^\perp \cap H \rangle : i \in \{1, \ldots, [d]\}\}$.

(c) *We have that* $z_i \cap z_j$ *is a* $(d-2)$*-dimensional subspace of* g.

Proof. By Proposition 6.8 (d) and $Y \neq Z$ (as G and H are disjoint), exactly $[d]$ elements of Z meet G in dimension $d-1$. This shows (a). By Proposition 6.8 (b), $\dim(z_i \cap z_j) < d-1$ for $i \neq j$. Hence, each hyperplane π_i of G lies in exactly one element z_i, and z_i satisfies $z_i \subseteq \pi_i^\perp$. Since (Y, Z) is a cross-intersecting EKR set, all z_j meet H in at least a point. Since $\pi_i \subseteq G$ and $G \cap H = \emptyset$, we see that $\pi_i^\perp \cap H$ is a point. Hence,

$$\{z_i : i \in \{1, \dots, [d]\}\} = \{\langle \pi_i, \pi_i^\perp \cap H \rangle : i \in \{1, \dots, [d]\}\}.$$

This shows (b). We have $z_i \cap z_j \subseteq G$ for $i \neq j$ as otherwise $\langle z_i \cap G, z_j \cap G, z_i \cap z_j \rangle$ would be a totally isotropic subspace of dimension at least $d+1$. As $z_i \cap G = \pi_i$ and $z_j \cap G = \pi_j$ are different hyperplanes of G, $z_i \cap z_j$ has dimension $d-2$. This shows (c). □

6.2.2.1 *The Parabolic Quadric* $Q(2d, q)$

Let s be a subspace of $PG(2d, q)$. We write $Y \subseteq s$ if all elements of Y are subspaces of s, $Y \cap s$ for all elements of Y in s, and $Y \setminus s$ for all elements of Y not in s.

Lemma 6.10. *Let* G *and* H *be disjoint generators of* $Q(2d, q)$. *Let* h *be the span of* G *and* H. *Then* $\mathcal{Q} := h \cap Q(2d, q)$ *is isomorphic to* $Q^+(2d-1, q)$.

Proof. The generators G and H are disjoint, hence $\mathcal{Q}$ is not degenerate. The hyperplane h obviously contains generators, hence $\mathcal{Q}$ is not isomorphic to $Q^-(2(d-1)+1, q)$. Therefore, the intersection $h \cap Q(2d, q)$ is isomorphic to $Q^+(2d-1, q)$. □

Lemma 6.11. *Let* (Y, Z) *be a cross-intersecting EKR set of* $Q(2d, q)$, d *even, of maximum size such that* $Y \neq Z$. *Let* $G \in Y$. *Let* $\tilde{Y}$ *be the set of the* $q^{\binom{d}{2}}$ *generators of* Y *disjoint to* G *(see Proposition 6.8).*

(a) *There exists a hyperplane* h *isomorphic to* $Q^+(2d-1, q)$ *such that* $G, \tilde{Y} \subseteq h$.

(b) *Let* $z \in Z$. *If* z *meets an element of* $\{G\} \cup \tilde{Y}$ *in a subspace of dimension* $d-1$, *then* $z \subseteq h$ *and all elements of* Z *disjoint to* z *are in* h.

(c) *If* $\tilde{G} \in Y$ *and* $\dim(\tilde{G} \cap H) = d-2$ *for an* $H \in \tilde{Y}$*, then* $\tilde{G}$ *and the* $q^{\binom{d}{2}}$ *generators disjoint to* $\tilde{G}$ *are in* h*.*

Proof. By Proposition 6.8, a generator $G \in Y$ is disjoint to $q^{\binom{d}{2}}$ generators of Y. Let $H \in \tilde{Y}$. By Lemma 6.10, the intersection of $h := \langle G, H\rangle$ with $Q(2d-1, q)$ is isomorphic to $Q^+(2d-1, q)$. We shall show $\tilde{Y} \subseteq h$.

Suppose to the contrary that there exists a generator $\tilde{H} \in \tilde{Y}$ not in h. We write

$$\begin{aligned} Z_A &= Z_{A,B} = \{z_i : i \in \{1, \ldots, [d]\}\} \\ &= \{\langle \pi_i, \pi_i^\perp \cap B\rangle : i \in \{1, \ldots, [d]\}\} \subseteq Z \end{aligned} \tag{6.1}$$

for $A, B \in Y$ disjoint in the notation of Lemma 6.9. In particular, notice that $Z_{G,H} = Z_{G,\tilde{H}}$ by Lemma 6.9 (a).

Set

$$\mathcal{P} := \{\tilde{h} \cap z_i : i \in \{1, \ldots, [d]\}\}.$$

By Lemma 6.9 (c), $|\mathcal{P}| = |Z_G| = [d]$. Furthermore, $Z_G \subseteq \eta$, so $\mathcal{P} \subseteq \tilde{H} \cap \eta$. Hence,

$$[d] = |\mathcal{P}| \leqslant |\tilde{H} \cap \eta| = [d-1].$$

This is a contradiction. Thus, $\tilde{Y} \subseteq h$. This proves (a).

To prove (b), suppose without loss of generality that z meets G in a subspace of dimension $d-1$. We have $z \in Z_{G,H}$ as (Y, Z) is a cross-intersecting EKR set, so $z \in h$. Let P be a point of G disjoint to z. By Lemma 6.9 (b), the generator $\tilde{z}$ defined as $\langle P, P^\perp \cap H\rangle$ is in $Z_{H,G}$. Then $\tilde{z} \subseteq h$. By Lemma 6.9 (c), $\tilde{z}$ is disjoint to z, since otherwise $\langle P, \tilde{z} \cap z, z \cap G\rangle$ would be a totally isotropic subspace of dimension $d+1$. Hence, $\langle z, \tilde{z}\rangle = h$. By (a), all elements of Z disjoint to z are in h. This shows (b).

Let $\tilde{G} \in Y$ with $\dim(H \cap \tilde{G}) = d-2$. Then there exists a $z \in Z_{H,G}$ with $H \cap \tilde{G} \subseteq z$. By (b) and $\dim(H \cap \tilde{G}) = d-2$, $\tilde{G} \subseteq h$. Let $\tilde{H} \in Y$ be disjoint to $\tilde{G}$. As (Y, Z) is a cross-intersecting EKR set, z meets $\tilde{H}$ in a point. Let $\tilde{z} \in Z_{\tilde{H},\tilde{G}}$ with z disjoint to $\tilde{z}$. By (b) and $\dim(H \cap z) = d-1$, $\langle z, \tilde{z}\rangle = h$. Hence, $\tilde{H} \subseteq h$. By (a), all elements of Y disjoint to $\tilde{G}$ are in h. This shows (c). □

Proposition 6.12. *Let* (Y, Z) *be a cross-intersecting EKR set of* $Q(2d, q)$ *of maximum size such that* $Y \cap Z \neq Y$. *Then there exists a hyperplane* h *such that* $Y, Z \subseteq h$.

Proof. Let $G \in Y$. In the view of Lemma 6.11, we find a hyperplane h that contains a generator $G \in Y$, a set Y_1 of $q^{\binom{d}{2}}$ generators of Y disjoint to G, and, by Proposition 6.8 and Lemma 6.11 (d), the set Y_2 of $\begin{bmatrix} d \\ 2 \end{bmatrix} q^{\binom{d-2}{2}}$ generators of Y which meet G in dimension $d-2$.

Suppose contrary to the assertion that there exists an element $\tilde{G} \in Y$ that is not in h. Then $\tilde{G}$ and the set Y_3 of $q^{\binom{d}{2}}$ generators of Y disjoint to $\tilde{g}$ lie in a second hyperplane $\eta' \neq \eta$ by Lemma 6.11. Lemma 6.11 (a) and (c) show that being contained in the same hyperplane is transitive for elements in Y, which are disjoint or meet in dimension $d-2$. By Lemma 6.11 (a) and $h \neq h'$, we have that $Y_1 \cap Y_3 = \emptyset$. By Lemma 6.11 (a), (c) and $h \neq h'$, we have that $Y_2 \cap Y_3 = \emptyset$. By Lemma 6.11 (a), (c) and $h \neq h'$, we have that $g \notin Y_3$, $\tilde{G} \notin Y_1 \cup Y_2$, and $G \neq \tilde{G}$.

Hence,

$$\begin{aligned} |Y| &\geqslant |Y_1| + |Y_2| + |Y_3| + 2 \\ &= 2\left(q^{\binom{d}{2}} + 1\right) + \begin{bmatrix} d \\ 2 \end{bmatrix} q^{\binom{d-2}{2}}. \end{aligned} \tag{6.2}$$

According to Theorem 6.3,

$$|Y| = \prod_{i=1}^{d-1} (q^i + 1). \tag{6.3}$$

By Lemma 2.1 and Lemma 2.8, (6.2) and (6.3) contradict each other. □

Remark 6.13. *Instead of the used counting argument, one could have used Lemma 6.11 (a), respectively, (c) and the (in thesis: unproven) fact that the subgraph induced by all Latin generators on the graphs* A_d, *respectively,* A_2 *of an hyperbolic space is connected.*

Theorem 6.14. *Let* (Y, Z) *be a cross-intersecting EKR set of* $Q(2d, q)$, *or* $W(2d-1, q)$ *with* q *even, of maximum size such that* $Y \cap Z \neq Y$. *Then* d *even and* $Y \cup Z$ *are the generators of a subgeometry isomorphic to* $Q^+(2d-1, q)$.

Proof. First consider $Q(2d, q)$. By Proposition 6.8, then $Y \cap Z = \emptyset$. By Proposition 6.12, $Y, Z \subseteq h$ for some hyperplane h isomorphic to $Q^+(2d-1, q)$ if not $Y = Z$. Hence, (Y, Z) is a cross-intersecting set of $Q^+(2d-1, q)$ of maximum size. These sets were classified in Theorem 6.4.

The part of the assertion for $W(2d-1, q)$, q even, follows, since in this case $Q(2d, q)$ and $W(2d-1, q)$ are isomorphic. □

6.2.2.2 *The Symplectic Polar Space* $W(2d-1, q)$, d *even,* q *odd*

Similar to [66] we use the following property of $W(2d-1, q)$, d even (see [65, 1.3.6, 3.2.1, 3.3.1]).

Theorem 6.15. *Let ℓ_1, ℓ_2, ℓ_3 be three pairwise disjoint lines of $W(3, q)$, q odd. Then the number of lines meeting ℓ_1, ℓ_2, ℓ_3 is 0 or 2.*

Theorem 6.16. *Let (Y, Z) be a cross-intersecting EKR set of $W(2d-1, q)$, d even, q odd, of maximum size. Then $Y = Z$.*

Proof. Suppose to the contrary that $Y \neq Z$. By Proposition 6.8, then $Y \cap Z = \emptyset$. By Proposition 6.8, we can find two disjoint generators G and H in Y. Again by Proposition 6.8, there are exactly $q[d][d-1]/(q+1)$ generators in Y which meet G in a subspace of dimension $d-2$. The generator G has $[d][d-1]/(q+1)$ subspaces of dimension $d-2$. Hence, we find a subspace $\ell \subseteq G$ of dimension $d-2$ such that ℓ is contained in q elements of Y'. Since q is odd, there are at least three elements y_1, y_2, y_3 of Y through ℓ.

Consider the quotient geometry $\mathcal{W}_3$ of ℓ isomorphic to $W(3, q)$ and the projection of the elements of Y and Z onto $\mathcal{W}_3$ from ℓ. Since elements of Y do not meet each other in dimension $d-1$ by Proposition 6.8, y_1, y_2, y_3 are three disjoint lines in $\mathcal{W}_3$ after projection. The subspace $\ell^\perp \cap H$ has dimension 2, so we find a subspace $\tilde{\ell} \subseteq H$ of dimension $d-2$ disjoint to $\ell^\perp$. Let π_1, π_2, π_3 be subspaces of dimension $d-1$ in H with $\tilde{\ell} \subseteq \pi_1, \pi_2, \pi_3$. By Lemma 6.9 (b), we find $z_1, z_2, z_3 \in Z$ with $\pi_i \subseteq z_i$ for $i \in \{1, 2, 3\}$. By Lemma 6.9 (c), the pairwise meets of z_1, z_2, z_3 are contained in $\tilde{\ell}$. As we have that $\tilde{\ell}$ meets $\ell^\perp$ trivially, we have that z_1, z_2, z_3 are projected onto three disjoint lines on $\mathcal{W}_3$. These three lines have to meet the projections of y_1, y_2, and y_3, since (Y, Z) is a cross-intersecting EKR set. This contradicts Theorem 6.15. □

6.3 THE HERMITIAN POLAR SPACE $H(2d-1,q^2)$

In this section we rewrite the proof of Theorem 4.1 as a Hoffman bound proof. Thereby we modify it in a way such it also works for cross-intersecting EKR sets in $H(2d-1,q^2)$, d even.

Theorem 6.17. *Let (Y,Z) be a cross-intersecting EKR set of $H(2d-1,q^2)$ with $d>1$. Then*

$$\sqrt{|Y|\cdot|Z|} \leqslant \frac{n\lambda_b}{\lambda_b-k} \approx q^{d^2-2d+2},$$

where $n=\prod_{i=0}^{d-1}(q^{2i+1}+1)$,

$$\lambda_b = -q^{(d-1)^2} - \alpha\left(1-f_1\frac{1-c}{n}\right),$$
$$k = q^{d^2}+\alpha f_1\left(c+\frac{1-c}{n}\right),$$
$$f_1 = q^2[d]_{q^2}\frac{q^{2d-3}+1}{q+1},$$
$$c = \frac{q^2-q-1+q^{-2d+3}}{q^{2d}-1},$$
$$\alpha = \begin{cases} q^{d(d-1)}-q^{(d-1)^2} & \text{if } d \text{ odd,}\\ \frac{q^{d(d-1)}+q^{(d-1)^2}}{1-2(1-c)f_1/n} & \text{if } d \text{ even.}\end{cases}$$

Proof. Let $d>1$. Let A_d be the disjointness matrix as defined in Section 5.5. Consider the matrix A defined as

$$A = A_d - \alpha E_1 + \frac{\alpha f_1 c}{n}J + \alpha f_1\frac{1-c}{n}I.$$

CLAIM 1 Our first claim is that A is an extended weight adjacency matrix. By Section 5.5, it is clear that the entry (x,y) of E_1 equals $Q_{i,1}/n$ if x and y meet in codimension i. It was shown at the beginning of Chapter 4 that the following holds (note that the equations in Chapter 4 do not depend on d odd):

(a) $Q_{0,1} = f_1$,

(b) $Q_{d-1,1} = f_1 c$,

(c) $Q_{s,1} \geqslant f_1 c$ if $s < d$ by (4.1),

(d) $Q_{d,1} < 0$.

Hence, the entry (x, y) of the matrix A is 0 if $x = y$, it is less or equal to zero if $1 \leqslant \operatorname{codim}(x \cap y) \leqslant d-1$, and it is larger than 1 if x and y are disjoint. This shows that A is an extended weight adjacency matrix of the disjointness graph of generators.

CLAIM 2 Our second claim is that one of the second absolute largest eigenvalues of A is

$$-q^{(d-1)^2} - \alpha\left(1 - f_1\frac{1-c}{n}\right),$$

and that

$$k = q^{d^2} + \alpha f_1\left(c + \frac{1-c}{n}\right).$$

Notice that A is a linear combination of the A_is of the scheme, so the common eigenspaces of the A_is are the eigenspaces of A. Notice for the following that $c \in [0, 1]$ and $f_1, \alpha > 0$. By Theorem 1.23, the eigenvalues of A are

$$\begin{aligned}
&q^{d^2} + \alpha f_1\left(c + \frac{1-c}{n}\right) \text{ for } \langle \mathbf{j} \rangle,\\
&-q^{(d-1)^2} - \alpha\left(1 - f_1\frac{1-c}{n}\right) \text{ for } V_1,\\
&(-1)^r q^{(d-r)^2+r(r-1)} + \alpha f_1\frac{1-c}{n} \text{ for } V_r \text{ with } 1 < r < d,\\
&(-1)^d q^{d(d-1)} + \alpha f_1\frac{1-c}{n} \text{ for } V_d.
\end{aligned}$$

a simple calculation shows that

$$-q^{(d-1)^2} - \alpha\left(1 - f_1\frac{1-c}{n}\right) = (-1)^d\left(q^{d(d-1)} - \alpha f_1\frac{1-c}{n}\right)$$

is the second largest absolute eigenvalue. This proves our claim.

Now we can apply Proposition 1.12 with these values. Note that k has approximately size q^{d^2+d-2}, the second largest absolute eigenvalue λ_b has approximately size $q^{d(d-1)}$, and n has approximately size q^{d^2}. Therefore,

$$\frac{n\lambda_b}{\lambda_b - k}$$

has approximately size q^{d^2-2d+2}. □

Note that the normal adjacency matrix of the graph only yields approximately q^{d^2-d} as an upper bound, so this improves the standard bound significantly.

For the sake of completeness we want to mention all cross-intersecting EKR sets for $d = 2$. We will do this after providing a general geometrical result on maximal cross-intersecting EKR sets, where we call a cross-intersecting EKR set (Y, Z) *maximal* if there exists no generator x such that $(Y \cup \{x\}, Z)$ or $(Y, Z \cup \{x\})$ is a cross-intersecting EKR set.

Lemma 6.18. *Let* (Y, Z) *be a maximal cross-intersecting EKR set in a finite classical polar space of rank* d. *If two distinct elements* $y_1, y_2 \in Y$ *meet in a subspace of dimension* $d-1$, *then all elements of* Z *meet this subspace in at least a point.*

Proof. Assume that there exists a generator z which meets y_1 and y_2 in points P, Q not in $y_1 \cap y_2$. Then $\langle P, Q, y_1 \cap y_2\rangle$ is a totally isotropic subspace of dimension $d + 1$. Contradiction. □

Proposition 1.12 yields

$$\frac{(q+1)(q^3+1)}{q^2+1}$$

as an upper bound on cross-intersecting EKR sets of $H(3, q^2)$. This bound is not sharp as the following trivial result shows.

Theorem 6.19. *Let* (Y, Z) *be a maximal cross-intersecting EKR set of the polar space* $H(3, q^2)$ *with* $|Y| \geqslant |Z|$. *Then one of the following cases occurs:*

(a) The set Y *is the set of all lines of* $H(3, q^2)$, *and* $Z = \emptyset$. *Here* $|Y| \cdot |Z| = 0$.

(b) *The set* Y *is the set of all lines meeting a fixed line* ℓ *in at least a point, and* $Z = \{\ell\}$*. Here* $|Y| \cdot |Z| = (q^2+1)q+1$.

(c) *The set* Y *is the set of all lines on a fixed point* P*, and* $Y = Z$*. Here* $|Y| \cdot |Z| = (q+1)^2$.

(d) *The set* Y *is the set of lines meeting two disjoint lines* ℓ_1, ℓ_2*, and* $Z = \{\ell_1, \ell_2\}$*. Here* $|Y| \cdot |Z| = 2(q^2+1)$.

(e) *The set* Y *is the set of lines meeting three disjoint lines* ℓ_1, ℓ_2, ℓ_3*, and* Z *is the set of all* $q+1$ *lines meeting the lines of* Y*. Here* $|Y| \cdot |Z| = (q+1)^2$.

Proof. Suppose that (a) does not occur.

By Lemma 6.18, as soon as two elements of Y meet in a point P, then all elements of Z contain P. Hence, (b) occurs or at least 2 elements of Z meet in P. Hence, all elements of Y contain P by Lemma 6.18. This is case (c).

So assume that Y and vice-versa Z only consist of disjoint lines. If there are two lines $\ell_1, \ell_2 \subseteq Z$, then there are q^2+1 (disjoint) lines L meeting ℓ_1 and ℓ_2 in a point (hence $|Y| \leqslant q^2+1$). If more than $q+1$ of these lines meet ℓ_1 (hence $|Y| > q+1$), then Z contains at most two lines, since in $H(3, q^2)$ exactly $q+1$ lines meet 3 pairwise disjoint lines in a point. This yields (d). If $|Z| \geqslant 3$, then $|Y| \leqslant q+1$ by the previous argument. We may assume $|Y| \geqslant 3$. Then it is well-known that there are exactly $q+1$ lines meeting the $q+1$ lines of Y. Hence, we can add these lines and then Z is maximal. This yields (e). □

The author tried to prove that the largest cross-intersecting EKR set of $H(5, q^2)$ is the unique EKR of maximum size given in [66], but aborted this attempt after he got lost in too many case distinctions. This EKR set of all generators meeting a fixed generator in at least a line is the largest cross-intersecting EKR set known to the author and has size q^5+q^3+q+1. The largest example known to the author for $H(7, q^2)$ is the following.

Example 6.20. *Let* G *be a generator of* $H(7, q^2)$*. Let* Y *be the set of all generators that meet* G *in at least a line. Let* Z *be the set of all generators that meet* G *in at least a plane. Then* (Y, Z) *is a cross-intersecting EKR set.*

Proof. A generator of $H(7, q^2)$ is a 4-space. A plane and a line of a 4-space meet pairwise in at least a point. Hence, (Y, Z) is a cross-intersecting EKR set. □

In this example Y has

$$1 + q + q^3 + q^4 + q^5 + q^6 + q^7 + 2q^8 + q^{10} + q^{12}$$

elements, Z has

$$1 + q + q^3 + q^5 + q^7$$

elements, so in total the cross-intersecting EKR set has size

$$\sqrt{|Y| \cdot |Z|} \approx q^{19/2}.$$

The bound given in Theorem 6.17 for this case is approximately q^{10}. For $H(2d-1, q^2)$, $d > 4$, the largest example known to the author is the EKR set of all generators on a fixed point. The author assumes that the largest known examples are also the largest examples.

6.4 CONCLUDING REMARKS

We summarize our results in the following table. We only list the cases, where cross-intersecting EKR sets of maximum size are not necessarily EKR sets. The table includes the size of the largest known example if it is not known if the best known bound does not seem to be tight.

Polar Space	The Maximum Size of $\sqrt{\lvert Y\rvert \cdot \lvert Z\rvert}$	The Largest (known) Examples	Th.
$Q^+(2d-1,q)$, d odd	$n/2$	Y latins, Z greeks	6.4
$Q(2d,q)$, d odd	$(q+1)\cdot\ldots\cdot(q^{d-1}+1)$	Y latins and Z greeks of a $Q^+(2d+1,q)$, or $Y=Z$ EKR set	6.14
$W(2d-1,q)$, d odd, q even	$(q+1)\cdot\ldots\cdot(q^{d-1}+1)$	see $Q(2d,q)$	6.14
$H(3,q^2)$	q^3+q+1	$Z=\{\ell\}$, Y all lines meeting ℓ	6.19
$H(5,q^2)$	$\lessapprox q^5$	largest EKR set, size $\approx q^5$	6.17
$H(7,q^2)$	$\lessapprox q^{10}$	Example 6.20, size $\approx q^{19/2}$	6.17
$H(2d-1,q^2)$, $d>1$	$\lessapprox q^{(d-1)^2+1}$	all generators on a point, size $\approx q^{(d-1)^2}$	6.17

7 PARTIAL SPREADS OF HERMITIAN POLAR SPACES

In this chapter we improve the best known bounds for $\{i\}$-cliques of generators of $H(2d-1, q^2)$ for large even d. In particular, the result improves some of the best known upper bounds for partial spreads of $H(2d-1, q^2)$. The result is a combination of the extensive work of the author with the eigenvalue matrices of the dual polar graphs while he was reading old lecture notes by Chris Godsil and discussing various clique bounds with John Bamberg in Perth, Western Australia. The lecture notes mention the multiplicity bound of Proposition 1.15. At the same time the author worked with an association scheme of $H(2d-1, q^2)$ where one multiplicity is very small (and therefore the bound is good). The proof of the result itself is trivial, but definitely not folklore for $d > 1$ as for example [20] fails to mention it.

An $\{d\}$-clique in the dual polar graph of $H(2d-1, q^2)$ is called a *partial spread*, since it is a set of pairwise disjoint generators. The first complete survey on spreads of polar spaces was done by Thas [73] in 1981. Later this problem was generalized to the study of partial spreads on polar spaces. The best result known to the author on the maximum size of partial spreads in $H(2d-1, q^2)$, d even, is due to De Beule, Klein, Metsch, and Storme [20].

The problem of the maximum size of partial spreads is a special case of the problem of the maximum size of *constant distance codes* of generators in $H(2d-1, q^2)$. Constant distance codes are of particular importance for random network coding as introduced in [55]. We refer to [56] for the general concept of constant distance codes of subspaces. For the purpose of this thesis, constant distance codes are just $\{i\}$-cliques in a distance-regular graph, i.e. for generators of Hermitian polar spaces constant distance codes are sets of subspaces which pairwise intersect in codimension i. Particularly, partial spreads are constant distance codes with $i = d$. The only non-trivial upper bounds known to the author on these sets for general i were provided in the PhD thesis of Vanhove [78].

Notice that partial spreads fit very well into the investigation of EKR sets with pairwise intersections in points, since it is its dual problem: an $\{d\}$-coclique of $H(2d-1, q^2)$ is an EKR set of $H(2d-1, q^2)$.

7.1 RELATED RESULTS

Recall that the Hermitian polar space $\mathsf{H}(2d-1, q^2)$ posses

$$\prod_{i=1}^{d}(q^{2i-1}+1)$$

generators and

$$\frac{(q^{2d-1}+1)(q^{2d}-1)}{q^2-1}$$

points. The number of generators on a point of $\mathsf{H}(2d-1, q^2)$ equals the number of generators of $\mathsf{H}(2d-3, q^2)$. A generator of $\mathsf{H}(2d-1, q^2)$ contains $(q^{2d}-1)/(q^2-1)$ points. Let Y be a (partial) spread of $\mathsf{H}(2d-1, q^2)$. Using the given combinatorial properties, double counting pairs (P, G) with $P \in G$ and $G \in Y$ yields

$$|Y| \leqslant q^{2d-1}+1 \tag{7.1}$$

with equality if and only if Y is a spread. This bound is never reached for $d > 1$. In some sense this bound corresponds to the sphere packing bound for codes if we consider all generators on a point as a sphere. One obtains the same bound by applying Proposition 1.9 on the graph with adjacency matrix $A = A_1 + A_2 + \ldots + A_{d-1}$ (with A as the extended weight matrix).

In the following we list the previous results on (partial) spreads in $\mathsf{H}(2d-1, q^2)$ known to the author that improve bound (7.1).

Theorem 7.1 (De Beule, Klein, Metsch, Storme [20])**.** *Let Y be a partial spread of $\mathsf{H}(3, q^2)$. Then*

$$|Y| \leqslant \frac{1}{2}(q^3+q+2).$$

In particular, this bound is sharp for $q = 2, 3$.

Theorem 7.2 (De Beule, Klein, Metsch, Storme [20])**.** *Let Y be a partial spread of $\mathsf{H}(2d-1, q^2)$, $d > 2$ even. Then*

$$|Y| \leqslant q^{2d-1} - q^{3d/2}(\sqrt{q}-1).$$

The following theorem is stated for the more general concept of near polygons in [78]. The Hermitian polar space $\mathsf{H}(2d-1, q^2)$ is a regular near 2d-polygon of order (q^2, q), so we will only provide this theorem for this particular case.

Theorem 7.3 (Vanhove [78, Theorem 6.4.10]). *Let* Y *be a set of generators of* $\mathsf{H}(2d-1, q^2)$ *such that all elements of* Y *pairwise intersect in codimension* i *odd. Then*

$$|Y| \leqslant 1 + q^i.$$

Our result is the following:

Theorem 7.4. *Let* Y *be a set of generators of* $\mathsf{H}(2d-1, q^2)$*,* $d > 1$*, such that all elements of* Y *pairwise intersect in codimension* i*. Then*

$$|Y| \leqslant q^{2d-1} - q\frac{q^{2d-2}-1}{q+1}.$$

In particular, this is a bound on the maximum size of partial spreads in $\mathsf{H}(2d-1, q^2)$.

If i is even, then this bound seems to be better than a any linear programming approach using Proposition 1.14. In many cases, it is also far better than any simple counting argument. If i is odd, then linear programming in form of Theorem 7.3 is far better.

First we will compare the case that Y is a partial spread (so $i = d$), d even, to the previous results in the following table.

d even	q	Best known bound	Theorem
2	2	6	7.1, 7.4
2	4	25	[17]
2	$\neq 4$	$\frac{1}{2}(q^3+q+2)$	7.1
4	2, 3	$q^{2d-1} - q\frac{q^{2d-2}-1}{q+1}$	7.4 (new)
4	> 3	$q^{2d-1} - q^{3d/2}(\sqrt{q}-1)$	7.2
> 4		$q^{2d-1} - q\frac{q^{2d-2}-1}{q+1}$	7.4 (new)

These bounds are sharp for $H(3,4)$ [28] and $H(3,9)$ [29]. They are not sharp for $H(3,16)$ [17].[1] For all other cases the tightness of these bounds seems to be unknown.

For d odd a sharp upper bound of $q^d + 1$ on the maximum size of partial spreads of $H(2d-1, q^2)$ was proven by Vanhove [77]. Examples reaching this bound were given by Aguglia, Cossidente, Ebert for $d = 3$ [1], and by Luyckx for $d > 3$ odd [58].

Now we will discuss the general case. If i is odd, then Theorem 7.3 gives a better bound for all i. In particular, for $i = 1$ it is well-known that the largest example is the set of all $q+1$ generators on a fixed subspace of rank $d-1$ (see Lemma 5.5). According to [39, Remark 4] there exists a constant-rank distance code with $i = 2$ and $q = 2$ in $H(2d-1, q^2)$ of size

$$\frac{q^{2d}-1}{q^2-1}.$$

For $q = 2$ this is one less than the bound of Theorem 7.4. Therefore, Theorem 7.4 is nearly sharp. In particular, in the case $q = 2$, $i = 2$ the bound is sharp (see above) and $d = 3$ as shown in a yet unpublished result by Maarten De Boeck.

A similar application of Godsil's bound for near polygons known to the author is the upper bound $s^5 - s^3 + s - 1$ on the size of partial distance-2 ovoids in the generalized hexagon with parameter (s, s^3) by Coolsaet and Van Maldeghem [18]. As in [78, Theorem 6.4.10] this bound and the new result could be stated in an unified way for many so-called *near polygons* with parameters $(s, t, c_1, \ldots, c_d)$ with $t \geqslant s^2$ and $c_i = (s^{2i}-1)/(s^2-1)$. Unfortunately, according to [24, Theorem 3.4] there are no other near polygons with these parameters for $d > 2$. To illustrate this point, we will sketch a more general proof for generalized polygons in Appendix A.

1 This result by Cimráková and Fack is due to an intelligent computer search. A purely combinatorial proof can be found in the PhD thesis of Beukemann [6].

7.2 PROOF OF THEOREM 7.4

By Theorem 1.22, the dual polar graph $H(2d-1, q^2)$ has the multiplicities

$$f_d = q^{2d}\frac{q^{1-2d}+1}{q+1} = q^{2d-1} - q\frac{q^{2d-2}-1}{q+1}.$$

By Theorem 1.23, our eigenvalues satisfy

$$\frac{P_{di}}{P_{0i}} = (-q)^{-i}.$$

By Lemma 1.6,

$$Q_{id} = \frac{P_{di}}{P_{0i}}Q_{0d} = f_d(-q)^{-i}.$$

Proof of Theorem 7.4. Let Y be a set of generators of $H(2d-1, q^2)$ such that the generators of Y are pairwise in relation R_0 or R_i, $i > 0$. By Proposition 1.15,

$$|Y| \leqslant f_d = q^{2d-1} - q\frac{q^{2d-2}-1}{q+1},$$

since

$$Q_{0d} = f_d \neq f_d(-q)^{-i} = Q_{id} \neq -1.$$

The assertion follows. □

8 THE MMS CONJECTURE FOR VECTOR SPACES

Recall Conjecture 3.8 which was proven by Chowdhury, Huang, Sarkis, Shahriari, and Sudakov [16, 47] for $n \geqslant 3k$. This chapter is devoted to the proof of the following result which is the result of a minor improvement of the technique used by Chowdhury, Sarkis, and Shahriari. Besides small q, the given theorem completely solves the MMS problem for vector spaces with weights on points.

Theorem 8.1. *Let V be an n-dimensional vector space over a finite field $\mathbb{F}_q$. Let $\mathcal{P}$ be the set of 1-dimensional subspaces of V. Let $f : \mathcal{P} \to \mathbb{R}$ be a weighting of the 1-dimensional subspaces such that $\sum_{P \in \mathcal{P}} f(P) = 0$. Let $2 \leqslant x \leqslant k$. If one of the following conditions is satisfied, then there are at least $\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$ k-dimensional subspaces with nonnegative weights.*

(a) $(x-1)n \geqslant (2x-1)k - x + 2$, $3k > n \geqslant 2k+2$, *and* $q \geqslant (x-1)! \cdot 2^{x+2}$,

(b) $(x-1)n \geqslant (2x-1)k - x + 1$, $3k - 1 > n \geqslant 2k+1$, *and* $q \geqslant (x-1)! \cdot 2^{2x+1}$,

(c) $n \geqslant 3k$ *or* $n = 2k$, *and* $q \geqslant 2$.

If equality holds and $n \geqslant 2k+1$, then the set of nonnegative k-dimensional subspaces is the set of all k-dimensional subspaces on a fixed 1-dimensional subspace.

This implies the following for $x = k$.

Corollary 8.2. *Let V be an n-dimensional vector space over a finite field $\mathbb{F}_q$. Let $\mathcal{P}$ be the set of 1-dimensional subspaces of V. Let $f : \mathcal{P} \to \mathbb{R}$ be a weighting of the 1-dimensional subspaces such that $\sum_{P \in \mathcal{P}} f(P) = 0$. Let $k \geqslant 2$. Then there exists a $q_0 \geqslant 2$ such that the following holds. If $n \geqslant 2k$, and $q \geqslant q_0$, then there are at least $\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$ k-dimensional subspaces with nonnegative weight. If equality holds and $n \geqslant 2k+1$, then the set of nonnegative k-dimensional subspaces is the set of all k-dimensional subspaces on a fixed 1-dimensional subspace.*

We shall extend the technique used by Chowdhury et al. to prove this. The purpose of the main theorem is to show that Conjecture 3.8 holds for $n \geqslant 2k$ if q is large. Often we will ignore minor improvements on the condition on q if this would decrease the readability of

x	2	3	4	5	6
$n \geqslant$	$3k$	$5k/2$	$7k/3$	$9k/4$	$11k/5$
$q \geqslant$	16	64	384	3072	30720

Table 2: The lower bounds on q in Theorem 8.1 for small x if $n \geqslant 2k+2$. Note that the case $x = 2$ is basically [16] where a more careful counting achieved $q \geqslant 2$ for $n \geqslant 3k+1$.

x	2	3	4	5	6
$k \leqslant$	2	4	6	8	10
$q \geqslant$	32	256	3072	49152	983040

Table 3: The lower bounds on q in Theorem 8.1 for small k if $n = 2k+1$. Note that there is an unpublished result by Chowdhury, Sarkis, and Shahriari that shows the result for $(n, k, q) = (5, 2, q)$, $q > 2$. Based on their unpublished notes the author proved the conjecture for $(n, k, q) = (5, 2, q)$.

the formulas. With the used techniques the lower bound on q can not be much better than $(x-1)!$, so the condition on q in Theorem 8.1 is reasonably good.

8.1 A BOUND ON PAIRWISE INTERSECTING SUBSPACES

One crucial ingredient of the result by Chowdhury et al. is [16, Lemma 3.6] which roughly says the following.

Lemma 8.3 ([16, Lemma 3.6])**.** *Let $n \geqslant 2k$. Let A and C be two k-dimensional subspaces of V. If $A \cap C$ is a 1-dimensional subspace, then the number of k-dimensional subspaces of V which non-trivially intersect both A and C but do not contain $A \cap C$ is at most*

$$\frac{1}{q^{n-3k}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

It is possible to generalize this statement and we shall do so in this section with Lemma 8.6. Lemma 8.6 is the main improvement of [16] while all the other results are merely technically necessary reformulations of the methods given by Chowdhury et al.

Definition 8.4. *Let Y be a set of subspaces of an n-dimensional vector space V. We say that a subspace M intersects Y badly if all $A \in Y$ satisfy* $\dim(A \cap M) = 1$ *and all $A, B \in Y$ with $A \neq B$ satisfy* $\dim(A \cap B \cap M) = 0$.

Definition 8.5. *We shall call Y a bad configuration if it is a set of k-dimensional subspaces of an n-dimensional vector space V such that all $C \in Y$ intersect $Y \setminus \{C\}$ badly.*

Lemma 8.6. *Let $1 < x \leqslant k$. Let $q \geqslant 3$. Let $n \geqslant 2k + \delta \geqslant 2k + 1$. Suppose $(x-1)n \geqslant (2x-1)k - x + \delta$. Let Y be a bad configuration of k-dimensional subspaces of a vector space V with $x = |Y|$. Then the number of k-dimensional subspaces of V which meet Y badly is at most*

$$x^2 \cdot 2^x \cdot q^{-\delta} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Proof. Let $Y = \{A_1, \ldots, A_x\}$ be a bad configuration. Let $\tilde{B}$ be a k-dimensional subspace of V which meets Y badly. Define S as $\langle \tilde{B} \cap A_i : 1 \leqslant i \leqslant x \rangle$. By the definition of $\tilde{B}$, the subspace S meets Y badly. The subspace S has at most dimension x, since it is spanned by x vectors, and S has at least dimension 2, since it intersects Y badly (i.e. the subspaces $S \cap A_i$ are pairwise disjoint). Let m, $2 \leqslant m \leqslant x$, be the dimension of S. We shall provide upper bounds for the number of choices for S for given m in Part 1. Then, in Part 2, we will bound the number of choices to extend a given S to a k-dimensional subspace of V.

PART 1: THE NUMBER OF CHOICES FOR A BADLY INTERSECTING m-DIMENSIONAL SUBSPACE. **Case $x = m \geqslant 2$.** We have at most $[k]^x$ choices for the 1-dimensional intersections of S with $A_1, \ldots, A_x$, since any A_i contains $[k]$ 1-dimensional subspaces.

Case $x > m \geqslant 2$. Put $M = \{S \cap A_i : 1 \leqslant i \leqslant x\}$. By assumption, S has dimension m, so there exists a set $B \subseteq M$ with m elements such that $\langle B \rangle = S$. Hence, we can choose S in m steps by choosing B. More formally, define the sets B_i, $2 \leqslant i \leqslant m$, as follows. Put $B_1 = \{S \cap A_1\}$. For $B_1, \ldots, B_i$ given, let j_0 the smallest number such that the meet of $\langle B_i \rangle$ and A_{j_0} is trivial. Set

$$B_{i+1} = B_i \cup \{S \cap A_{j_0}\}.$$

Define M_i as $\{\langle B_i \rangle \cap A_j : \langle B_i \rangle \cap A_j$ non-trivial$\}$.

Since $m < x$, we have $|M_i| \geqslant |M_{i-1}| + 2$ for one i. Let i_0 be the smallest i where this occurs. Let A_j, A_{i_0} be two of the elements of Y which meet $\langle B_{i_0-1} \rangle$ trivially, but $\langle B_{i_0} \rangle$ non-trivially. Notice that $i_0 \in \{2, \ldots, m\}$ and $j \in \{i_0 + 1, \ldots, m\}$. In the following we are going to double count the tuples $(i_0, j, B_1, \ldots, B_m = B, S)$ in accordance with the given defintions.

For given $\{B_1, \ldots, B_{i-1}\}$, $i \neq i_0$, we have at most $[k]$ choices for B_i, since any A_i contains $[k]$ 1-dimensional subspaces. Hence, we have at most $[k]^{m-1}$ choices for all $\{B_i : 1 \leqslant i \leqslant m, i \neq i_0\}$.

We have at most $m - 1 \leqslant x - 2$ choices for i_0. We have at most $x - i_0 - 1$ choices for j for given i_0 as by construction all elements of $A_1, \ldots, A_{i_0-1}$ meet $\langle B_{i_0} \rangle$ non-trivially. Therefore, we have at most $\binom{x-1}{2}$ choices for the pair (i_0, j). By our choice of i_0, $A_{i_0} \cap \langle B_{i_0} \rangle$ is a subspace of $\langle B_{i_0}, A_j \rangle$. By $k = \dim(A_j) = \dim(A_{i_0})$, we have

$$\begin{aligned}
&\dim(\langle B_{i_0}, A_j \rangle \cap A_{i_0}) \\
&\leqslant \dim(\langle B_{i_0}, A_j \rangle) + \dim(A_{i_0}) - \dim(\langle A_j, A_{i_0} \rangle) \\
&\leqslant (\dim(\langle B_{i_0} \rangle) + \dim(A_j) - 1) + \dim(A_j) - (2\dim(A_j) - 1) \\
&= \dim(\langle B_{i_0} \rangle) \leqslant m.
\end{aligned}$$

Therefore, we have at most $[m]$ choices to extend B_{i_0-1} to B_{i_0} by choosing a 1-dimensional subspace in A_{i_0}. For given $B = B_m$, S is uniquely determined by $S = \langle B \rangle$. Hence, we have $\binom{x-1}{2}[k]^{m-1}[m]$ choices for S for given $(i, j_0, B_1, \ldots, B_m)$.

On the other hand, for S given, B, B_i and therefore i_0 and j are uniquely determined by their definitions.

PART 2: THE NUMBER OF CHOICES FOR A k-DIMENSIONAL SUBSPACE ON A GIVEN m-DIMENSIONAL SUBSPACE. For given S have we have $\begin{bmatrix} n-m \\ k-m \end{bmatrix}$ choices for a k-dimensional subspace through S. So if $m = x$, then we have at most

$$[k]^x \begin{bmatrix} n-x \\ k-x \end{bmatrix} \leqslant 2^{x+1} q^{x(k-1)+(n-k)(k-x)} \tag{8.1}$$

choices for B by Corollary 2.1. If $m < x$, then we have at most

$$[k]^{m-1}\binom{x-1}{2}[m]\begin{bmatrix} n-m \\ k-m \end{bmatrix} \leqslant \binom{x-1}{2} \cdot 2^{m+1} q^{(m-1)k+(n-k)(k-m)} \tag{8.2}$$

choices for B by Corollary 2.1.

By (8.2) and $n \geqslant 2k+1$ we find that the choices for B for which the dimension of S is less than x is at most

$$\begin{aligned}
&\sum_{m=2}^{x-1}\binom{x-1}{2} \cdot 2^{m+1} q^{(m-1)k+(n-k)(k-m)} \\
&\leqslant \binom{x-1}{2}\left(\sum_{m=2}^{x-1} 2^{m+1}\right) \max_{m=2,\ldots,x-1} q^{(m-1)k+(n-k)(k-m)} \\
&\leqslant \binom{x-1}{2} \cdot 2^{x+1} \max_{m=2,\ldots,x-1} q^{(m-1)k+(n-k)(k-m)} \\
&= \binom{x-1}{2} \cdot 2^{x+1} q^{k+(n-k)(k-2)} \\
&= (x-1)(x-2) \cdot 2^{x} q^{k+(n-k)(k-2)}
\end{aligned}$$

Hence an upper bound for this number and (8.1) is, by $(x-1)n \geqslant (2x-1)k - x + \delta$, $n \geqslant 2k+\delta$, and $(x-1)(x-2)+x \leqslant x^2$ for $x \geqslant 2$,

$$x^2 \cdot 2^x q^{\max(k+(n-k)(k-2),x(k-1)+(n-k)(k-x))} \leqslant x^2 \cdot 2^x q^{-\delta+(n-k)(k-1)}. \tag{8.3}$$

Applying the inequality $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} > q^{(k-1)(n-k)}$ to (8.3) shows the assertion. □

8.2 AN EIGENVALUE TECHNIQUE

In this section we restate the arguments used in Section 3 of [16]. We include proofs for the results as our results extend results of [16] in some way. Before we do this we want to give some context to the used eigenvalue technique.

Let V be a vector space over $\mathbb{F}_q$. Let $f : \mathcal{P} \to \mathbb{R}$ a weighting of $\mathcal{P}$ with $\sum_{P\in\mathcal{P}} f(P) = 0$. We suppose $f \not\equiv 0$ throughout this section, since the case $f \equiv 0$ is trivial for Theorem 8.1. We say that two subspaces R and S are *incident* if $S \subseteq R$ or $R \subseteq S$. Define the *incidence matrix* W_{ij} as the matrix whose rows are indexed by the i-dimensional subspaces of V, whose columns are indexed by the j-dimensional subspaces of V, by

$$(W_{ij})_{RS} = \begin{cases} 1 & \text{if } S \text{ is incident with } R, \\ 0 & \text{otherwise.} \end{cases}$$

Let A_i be the distance-i-adjacency matrix of the Grassman graph (see Subsection 1.5.2). We write b_S for the weight of a k-dimensional subspace S of V (i.e. $b_S = \sum_{P\in S} f(P)$). By the definition of the weight of S, clearly $b = W_{k1} f$ holds if we consider $f = (f(P))_{P\in\mathcal{P}}$ as a vector indexed by the 1-dimensional subspaces P of V and $b = (b_S)_S$ as a vector indexed by the k-dimensional subspaces S of V. By Theorem 1.18, $b = W_{k1} f$ is an eigenvector of A_i and lies in the eigenspace V_1.

Lemma 8.7. *Let A_i be the distance-i adjacency matrix of the k-dimensional subspaces of V. Let b be the weight vector of the k-dimensional subspaces of V. Then b is an eigenvector of A_i with eigenvalue*

$$\begin{bmatrix} n-k-1 \\ i \end{bmatrix} \begin{bmatrix} k-1 \\ i \end{bmatrix} q^{(i+1)i} - \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k-1 \\ i-1 \end{bmatrix} q^{i(i-1)}.$$

For a given k-dimensional subspaces C we have

$$\sum_{\dim(S\cap C)=k-i} b_S = (A_i b)_C =$$
$$\left(\begin{bmatrix} n-k-1 \\ i \end{bmatrix} \begin{bmatrix} k-1 \\ i \end{bmatrix} q^{(i+1)i} - \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k-1 \\ i-1 \end{bmatrix} q^{i(i-1)} \right) b_C,$$

which makes these eigenvalues very useful. In particular, for A_{k-1} we get the following result. Notice that this number was directly calculated by Chowdhury et al. in [16, Equation (3.15)] and that we adopted their presentation of the formula.

Lemma 8.8. *Let* $n \geqslant 2k+1$*. Let* C *be a* k*-dimensional subspace of* V*. Then we have* $\sum_{\dim(S \cap C)=1} b_S =$

$$\left(q^{k(k-1)}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} - q^{(k-1)(k-2)}[k-1]\begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix}\right) b_C.$$

Let A be one of the k-dimensional subspaces with $b_A = \max b_S$ (i.e. a k-dimensional subspace with the highest weight). The idea of this section is to reach a situation where we can apply Lemma 8.6 on a large bad configuration. We shall do this in several steps. In Lemma 8.10 we show that we are able to find a lot of nonnegative k-dimensional subspaces which intersect A in a 1-dimensional subspace and have a weight of nearly b_A for large q. Lemma 8.11 then shows that many of these nonnegative k-dimensional subspaces pairwise intersect in exactly a 1-dimensional subspace, which leads to a situation where we can apply Lemma 8.6.

Lemma 8.9. *Let* $n \geqslant 2k+1$*. Let* A *denote a highest weight* k*-dimensional subspace of* V*. Let* C *be a nonnegative* k*-dimensional subspace of* V*. Then at least*

$$\left(1 - \frac{3}{q^{n-2k+1}}\right)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\frac{b_C}{b_A}$$

nonnegative k*-dimensional subspaces intersect* C *in exactly a* 1*-dimensional subspace.*

Proof. By Lemma 8.8, $\sum_{\dim(S \cap C)=1} b_S =$

$$\left(q^{k(k-1)}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} - q^{(k-1)(k-2)}[k-1]\begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix}\right) b_C.$$

Each b_S is less than or equal to b_A which yields at least

$$\left(q^{k(k-1)}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} - q^{(k-1)(k-2)}[k-1]\begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix}\right) \frac{b_C}{b_A}$$

nonnegative k-dimensional subspaces that intersect C in exactly a 1-dimensional subspace. As $n \geqslant 2k+1$, we have

$$\begin{aligned} &\frac{q^{(k-1)(k-2)}[k-1]\begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix}}{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}} \\ &= q^{(k-1)(k-2)}[k-1]\frac{q^{k-1}-1}{q^{n-k+1}-1}\prod_{i=1}^{k-2}\frac{q^{n-k-i}-1}{q^{n-i}-1} \\ &< q^{(k-1)^2}q^{-n+2k-2}q^{-(k-2)k} \\ &\leqslant \frac{1}{q^{n-2k+1}}. \end{aligned} \tag{8.4}$$

Then Lemma 2.3 (with $a = k$) shows the assertion. □

Lemma 8.10. *Let* $n \geqslant 2k+1$. *Let* c *be a real number with* $3 \leqslant c \leqslant q$. *Let* A *denote a highest weight* k*-dimensional subspace of* V. *Let* C_i *denote the* i*-th highest weight* k*-dimensional subspace of* V *such that* $\dim(A \cap C_i) = 1$. *Suppose* $i \leqslant \frac{c-3}{q}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + 1$, *and suppose that there are at most* $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ *nonnegative* k*-dimensional subspaces of* V, *then* b_{C_i}, *the weight of* C_i, *satisfies*

$$b_{C_i} > \left(1 - \frac{c}{q}\right) b_A$$

Proof. By Lemma 8.8 and $b_{C_i} \leqslant b_A$, we have

$$\sum_{j \geqslant i} b_{C_j} = \sum b_{C_j} - \sum_{j<i} b_{C_j} \geqslant b_A \cdot \left(q^{k(k-1)}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} - q^{(k-1)(k-2)}[k-1]\begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix} - i + 1\right).$$

We suppose that we have at most $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ nonnegative k-dimensional subspaces. Hence, we find

$$b_{C_i} \geqslant \frac{q^{k(k-1)}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} - q^{(k-1)(k-2)}[k-1]\begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix} - i + 1}{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}}. \tag{8.5}$$

By hypothesis $i \leqslant \frac{c-3}{q}\begin{bmatrix}n-1\\k-1\end{bmatrix}+1 \leqslant \left(\frac{c}{q}-\frac{3}{q^{n-2k+1}}\right)\begin{bmatrix}n-1\\k-1\end{bmatrix}+1$, $n \geqslant 2k+1$, and (8.4), we have

$$\begin{aligned} & q^{(k-1)(k-2)}[k-1]\begin{bmatrix}n-k-1\\k-2\end{bmatrix}+i-1 \\ & \leqslant \left(\frac{c}{q}-\frac{2}{q^{n-2k+1}}\right)\begin{bmatrix}n-1\\k-1\end{bmatrix}. \end{aligned} \tag{8.6}$$

By Lemma 2.3,

$$q^{k(k-1)}\begin{bmatrix}n-k-1\\k-1\end{bmatrix} \geqslant \left(1-\frac{2}{q^{n-2k+1}}\right)\begin{bmatrix}n-1\\k-1\end{bmatrix}.$$

Hence, (8.6) and (8.5) yield assertion. □

Lemma 8.11. *Let $n \geqslant 2k+\delta \geqslant 2k+1$. Let c be a real number with $3 \leqslant c \leqslant q$. Let $q \geqslant 3$. Let A denote a highest weight k-dimensional subspace of V. Let C_i denote the i-th highest weight k-dimensional subspace of V such that $\dim(A \cap C_i) = 1$. Let I be a subset of $\{1, \ldots, \lfloor \frac{c-3}{q}\begin{bmatrix}n-1\\k-1\end{bmatrix}+1 \rfloor\}$ with $|I| \leqslant k-1$. Set $x = |I|+1$. Set $M := \{A\} \cup \{C_i : i \in I\}$. Suppose that there are at most $\begin{bmatrix}n-1\\k-1\end{bmatrix}$ nonnegative k-dimensional subspaces. Then we have the following.*

(a) At least

$$\left(1-\frac{(x-1)c}{q}-\frac{3x}{q^{n-2k+1}}\right)\begin{bmatrix}n-1\\k-1\end{bmatrix}$$

nonnegative k-dimensional subspaces intersect each element of M in exactly a 1-dimensional subspace.

(b) Suppose that M is a bad configuration. Suppose $x > 1$ with $(x-1)n \geqslant (2x-1)k - x + \delta$. Then there exist $S, R \in M$ such that the 1-dimensional subspace $S \cap R$ lies in at least

$$\left(1-\frac{(x-1)c}{q}-\frac{3x}{q^{n-2k+1}}-x^2 \cdot 2^x q^{-\delta}\right)\frac{\begin{bmatrix}n-1\\k-1\end{bmatrix}}{\binom{x}{2}}$$

nonnegative k-dimensional subspaces.

Proof. We write $M = \{M_1, \ldots, M_x\}$ with $M_1 = A$.

First we shall show by induction on $x \geqslant 1$ that at least

$$\left(1 - \frac{(x-1)c}{q} - \frac{3x}{q^{n-2k+1}}\right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \tag{8.7}$$

nonnegative k-subspaces intersect all elements of $\{M_1, \ldots, M_x\}$ in exactly a 1-dimensional subspace.

For A we find by applying Lemma 8.9 that A meets at least

$$\left(1 - \frac{3}{q^{n-2k+1}}\right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \tag{8.8}$$

nonnegative k-dimensional subspaces in a 1-dimensional subspace. This shows (8.7) for $x = 1$.

Now suppose $x > 1$. By hypothesis, there are at most $\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$ nonnegative k-dimensional subspaces. So by Lemma 8.9, Lemma 8.10, (8.7), and the sieve principle, we find that at least

$$\begin{aligned} &\left(1 - \frac{(x-1)c}{q} - \frac{3x}{q^{n-2k+1}}\right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\ +&\left(1 - \frac{c}{q} - \frac{3}{q^{n-2k+1}}\right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\ \geqslant &\left(1 - \frac{xc}{q} - \frac{3(x+1)}{q^{n-2k+1}}\right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \end{aligned} \tag{8.9}$$

nonnegative k-dimensional subspaces intersect all elements of the set $\{M_1, \ldots, M_x\}$ in exactly a 1-dimensional subspace. This shows (a).

By Lemma 8.6 and (8.7), we have that at least

$$\left(1 - \frac{(x-1)c}{q} - \frac{3x}{q^{n-2k+1}} - x^2 \cdot 2^x q^{-\delta}\right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \tag{8.10}$$

nonnegative k-dimensional subspaces intersect all elements of M in exactly a 1-dimensional subspace and contain a 1-dimensional subspace of the form $M_i \cap M_j$, $i \neq j$. As we have at most $\binom{x}{2}$ such 1-dimensional subspaces $M_i \cap M_j$, this shows (b). □

Lemma 8.12. *Let $n \geqslant 2k+1$. Let x be a number with $2 \leqslant x \leqslant k$. Let $q \geqslant (x-1)! \cdot 2^{x+2}$. Let A denote a highest weight k-dimensional subspace* *of* V*. Let* C_i *denote the* i*-th highest weight* k*-dimensional subspace of* V *such that* $\dim(A \cap C_i) = 1$*. Suppose that there are at most* $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ *nonnegative* k*-dimensional subspaces of* V*. Suppose that no* 1*-dimensional subspace is contained in more than* $\frac{3}{q}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ *nonnegative* k*-dimensional subspaces. Then* $\tilde{M}_x := \{A\} \cup \{C_i : i \leqslant \frac{(x-2)! \cdot 2^{x+1} - 3}{q}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + 1\}$ *contains a bad configuration* M *with* x *elements and* $A \in M$*.*

Proof. We shall prove our claim by induction on x. If $x = 1$, then $\{A\}$ is a bad x-configuration. If $x = 2$, then $\{A, C_1\}$ is a bad x-configuration.

Only the case $x > 2$ remains. Suppose that $\tilde{M}_x$ contains a bad x-configuration $M = \{M_1, \dots, M_x\}$ with $A \in M$ and $x \geqslant 2$. By Lemma 8.11 (a), at least

$$\alpha := \left(1 - \frac{(x-1)! \cdot 2^{x+1}}{q} - \frac{3x}{q^{n-2k+1}}\right)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

nonnegative k-dimensional subspaces intersect all elements of M in exactly a 1-dimensional subspace. By hypothesis, at most

$$\frac{3}{q}\binom{x}{2}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

nonnegative k-dimensional subspaces of V contain one of the $\binom{x}{2}$ 1-dimensional subspaces $M_i \cap M_j$, $i \neq j$. So the number of nonnegative k-dimensional subspaces which meet M badly is, by definition, by $q \geqslant (x-1)! \cdot 2^{x+2}$, and by $n - 2k + 1 \geqslant 2$, at least

$$\begin{aligned}
&\alpha - \frac{3}{q}\binom{x}{2}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\
&= \left(1 - \frac{(x-1)! \cdot 2^{x+1} + 3\binom{x}{2}}{q} - \frac{3x}{q^{n-2k+1}}\right)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\
&\geqslant \left(1 - \frac{(x-1)! \cdot 2^{x+1} + 3\binom{x}{2} + 3x/q}{q}\right)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\
&\geqslant \left(1 - \frac{(x-1)! \cdot 2^{x+1} + 3\binom{x}{2} + 1}{q}\right)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} =: \beta.
\end{aligned}$$

There are at most $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ nonnegative k-dimensional subspaces and, by Lemma 8.10 and $q \geqslant (x-1)! \cdot 2^{x+2}$, all elements of $\tilde{M}_{x+1}$ have nonnegative weight, so at least

$$\begin{aligned} &\beta + |\tilde{M}_{x+1}| - \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\ \geqslant &\left(\frac{(x-1)! \cdot 2^{x+2} - 3 - (x-1)! \cdot 2^{x+1} - 3\binom{x}{2} - 1}{q} \right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\ = &\left(\frac{(x-1)! \cdot 2^{x+1} - 4 - 3\binom{x}{2}}{q} \right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \end{aligned}$$

nonnegative k-dimensional subspaces meet M badly and are in $\tilde{M}_{x+1}$. For $x \geqslant 2$ this number is positive, so we will find a bad configuration of nonnegative k-dimensional subspaces in $\tilde{M}_{x+1}$. □

8.3 AN AVERAGING BOUND

In [16, Lemma 4.5] Chowdhury, Sarkis, and Shahriari apply a result by Beutelspacher (Theorem 3.7) on partial spreads of projective spaces. They do not fully state what their proof shows which is why we have to restate their result here in a bit more detail. We refer[1] to [16] for the complete argument.

Lemma 8.13. *If* $n = 2k + \delta$ *with* $0 \leqslant \delta < k$, *and* T *is a negative weight* k*-dimensional subspace, then there are at least*

$$\left(1 - \frac{2}{q}\right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

nonnegative k*-dimensional subspaces that have trivial intersection with* T.

Proof. The proof is as in [16, Lemma 4.5] with the exception of [16, Equation (4.51)]. By Lemma 2.3 and [16, Equation (4.51)], we have for $n = 2k + \delta$

$$\begin{aligned} |\mathcal{F}| &\geqslant q^{(k+\delta)(k-1)} \begin{bmatrix} n-k-\delta-1 \\ k-1 \end{bmatrix} \\ &= q^{(k+\delta)(k-1)} \geqslant \left(1 - \frac{2}{q}\right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \end{aligned}$$

1 One can find a preprint of [16] on the arXiv.

□

8.4 PROOF OF THEOREM 8.1

Proof of Theorem 8.1. We may assume that there are at most $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ nonnegative k-dimensional subspaces. We will also assume $2k < n < 3k$, since the remaining cases are covered in [16, Theorem 1.3] and [61] (see Subsection 3.2.4). Also notice that the theorem reuqires $x \geqslant 2$.

If there exists a 1-dimensional subspace P which is contained in $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ nonnegative k-dimensional subspaces, then we are done. Therefore, we can suppose that all 1-dimensional subspaces are contained in at least one k-dimensional subspace with negative weight.

Suppose there exists a 1-dimensional subspace P which is contained in more than $\frac{2}{q}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ nonnegative k-dimensional subspaces. There exists a negative k-dimensional subspace T on P, so there are at least

$$\left(1-\frac{2}{q}\right)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

nonnegative k-dimensional subspaces not on P by Lemma 8.13. Then there are more than $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}+1$ nonnegative k-dimensional subspaces which contradicts our assumption.

Therefore no 1-dimensional subspace is contained in more than $\frac{2}{q}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ nonnegative k-dimensional subspaces. Let A denote a heighest weight k-dimensional subspace of V. Let C_i denote the i-th highest weight k-dimensional subspace of V such that $\dim(A \cap C_i) = 1$. By Lemma 8.12, there exists a bad configuration in $\{A\} \cup \{C_i : i \leqslant \frac{(x-2)!\cdot 2^{x+2}-3}{q}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}+1\}$ with x elements. Hence, we can apply Lemma 8.11 (b) with

$$c = (x-2)! \cdot 2^{x+1}$$

which shows that we find a 1-dimensional subspace that is a subspace of at least

$$\left(1-\frac{(x-1)!\cdot 2^{x+1}}{q}-\frac{3x}{q^{n-2k+1}}-x^2\cdot 2^x\cdot q^{-\delta}\right)\frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}}{\binom{x}{2}} \qquad (8.11)$$

nonnegative k-dimensional subspaces where $\delta = 2$ in Case (a), and $\delta = 1$ in Case (b).

If $\delta = 2$, then the assumptions $q \geqslant (x-1)! \cdot 2^{x+2}$, $(x-1)n \geqslant (2x-1)k - x + 2$ (particularly, $x > 1$), and $n \geqslant 2k+2$ imply

$$\begin{aligned}
&\frac{(x-1)! \cdot 2^{x+1}}{q} \leqslant \frac{1}{2},\\
&\frac{3x}{q^{n-2k+1}} \leqslant \frac{3x}{q^3} \leqslant \frac{3x}{2^{3(x+2)}} \leqslant \frac{3}{256},\\
&x^2 \cdot 2^x \cdot q^{-\delta} \leqslant \frac{x^2 \cdot 2^x}{(2^{x+2})^2} \leqslant \frac{x^2}{2^{x+4}} \leqslant \frac{1}{8},\\
&\frac{q}{\binom{x}{2}} \geqslant \frac{(x-1)! \cdot 2^{x+3}}{x(x-1)} \geqslant 16,
\end{aligned}$$

so (8.11) is at least

$$\begin{aligned}
\left(1 - \frac{1}{2} - \frac{3}{256} - \frac{1}{8}\right) \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}}{\binom{x}{2}} &= \frac{93}{256\binom{x}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\\
&\geqslant \frac{93}{16q} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} > \frac{2}{q} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.
\end{aligned}$$

This contradicts our assumption that no 1-dimensional subspace P is contained in more than $\frac{2}{q}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ nonnegative k-dimensional subspaces. Hence, Part (a) of the theorem follows.

If $\delta = 1$, then the assumptions $q \geqslant (x-1)! \cdot 2^{2x+1}$, $(x-1)n \geqslant (2x-1)k - x + 1$, and $n \geqslant 2k+1$ imply Part (b) of the theorem with similar calculations. Here we have

$$\begin{aligned}
&\frac{(x-1)! \cdot 2^{x+1}}{q} \leqslant \frac{1}{4},\\
&\frac{3x}{q^{n-2k+1}} \leqslant \frac{3x}{q^3} \leqslant \frac{3x}{2^{3(2x+1)}} \leqslant \frac{3}{1024},\\
&x^2 \cdot 2^x \cdot q^{-\delta} \leqslant \frac{x^2 \cdot 2^x}{2^{2x+1}} \leqslant \frac{x^2}{2^{x+1}} \leqslant \frac{9}{16},\\
&\frac{q}{\binom{x}{2}} \geqslant \frac{(x-1)! \cdot 2^{2x+2}}{x(x-1)} \geqslant 32.
\end{aligned}$$

Then (8.11) is at least

$$\left(1-\frac{1}{4}-\frac{3}{1024}-\frac{9}{16}\right)\frac{\begin{bmatrix}n-1\\k-1\end{bmatrix}}{\binom{x}{2}} = \frac{189}{1024\binom{x}{2}}\begin{bmatrix}n-1\\k-1\end{bmatrix}$$
$$\geqslant \frac{189}{32q}\begin{bmatrix}n-1\\k-1\end{bmatrix} > \frac{2}{q}\begin{bmatrix}n-1\\k-1\end{bmatrix}.$$

□

8.5 DUALITY

For the sake of completeness we also mention the following simple exercise, which was (at least when the author first published it) not common knowledge among the experts on the topic.

Lemma 8.14. *Let* $n \geqslant 2k$*. If there are at least* α $(n-k)$*-dimensional subspaces with nonnegative weight, then there are at least* α k*-dimensional subspaces with nonnegative weight. Furthermore, the set of* k*-dimensional subspaces with nonnegative weights is isomorphic to a dual of the set of nonnegative* $(n-k)$*-dimensional subspaces.*

Proof. Let $\mathcal{H}$ be the set of hyperplanes of V. Define the weight function $g : \mathcal{H} \to \mathbb{R}$ by $g(H) = \sum_{P\in H} f(P)$. Define the g-weight of a k-dimensional subspace U by $g(U) = \sum_{U\subseteq H} g(H)$. Furthermore by $\sum_{P\in\mathcal{P}} f(P) = 0$,

$$\begin{aligned}
g(U) &= \sum_{U\subseteq H} g(H)\\
&= \sum_{U\subseteq H}\sum_{P\in H} f(P)\\
&= \sum_{U\subseteq H}\left(\sum_{P\in H\cap U} f(P)\right) + \left(\sum_{P\in H\setminus U} f(P)\right)\\
&= \sum_{P\in\mathcal{P}\cap U}([n-k]f(P) + \sum_{P\in\mathcal{P}\setminus U}[n-k-1]f(P))\\
&= q^{n-k-1}f(U).
\end{aligned}$$

Hence, we can consider the problem in the dual vector space of V (which is isomorphic to V) with g as the weight function on points (of the dual space). Then the assertion is obvious. $\square$

This shows that one only has to investigate the MMS problem for $n \geqslant 2k$: If $n < 2k$, then $n > n + (n - 2k) = 2(n - k)$.

8.6 CONCLUDING REMARKS

The used argument is based on the observation that the only set of nonnegative k-dimensional subspaces which reaches the bound $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ seems to be the set of all generators on a fixed 1-dimensional subspace. This can no longer work for $n = 2k$, since one can construct another example of that size as follows. Fix a $(2k-1)$-dimensional subspace S, put the weight -1 on all 1-dimensional subspaces not in S and the weight $q^{2k-1}/[2k-1]$ on all 1-dimensional subspaces in S. Then exactly the $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ k-dimensional subspaces in S are the nonnegative ones, so this is a second example. This is in fact the only other example in this case (see below).

Conjecture 3.8 is wrong for $k < n < 2k$ as one can see by a similar example which we obtain by duality: Fix a $(n-1)$-dimensional subspace S, put the weight -1 on all 1-dimensional subspaces not in S, put the weight $q^{n-1}/[n-1]$ on all 1-dimensional subspaces in S. Then the nonnegative k-dimensional subspaces are exactly the k-dimensional subspaces in S. There are $\begin{bmatrix} n-1 \\ k \end{bmatrix}$ such subspaces, so $\begin{bmatrix} n-1 \\ k \end{bmatrix} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ for $k < n < 2k$ shows that Conjecture 3.8 does not hold in this range.

As the cases $n < 2k$ are covered by Lemma 8.14, it seems to be reasonable to conjecture the following.

(a) For $k < n < 2k$, the minimum number of nonnegative k-dimensional subspaces is $\begin{bmatrix} n-1 \\ k \end{bmatrix}$ with equality for the example given above,

(b) For $n = 2k$, the minimum number of nonnegative k-dimensional subspaces is $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ with equality for the two given example, i.e. either all nonnegative k-dimensional subspaces contain a fixed

1-dimensional subspace or all nonnegative k-dimensional subspaces are contained in a fixed $(n-1)$-dimensional subspace,

(c) For $n > 2k$, the minimum number of nonnegative k-dimensional subspaces is $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ with equality if and only if all nonnegative k-dimensional subspaces contain a fixed 1-dimensional subspace.

Notice that (b) is implied by the proof of [61, Theorem 3.1] and the classification of all Erdős-Ko-Rado sets of size $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. Chowdhury remarked[2] that this conjecture is the canonical generalization of a conjecture on the MMS problem for sets given in [4, 11] which was confirmed for small cases in [41]. Additionally, the author did a (non-exhaustive) computer search for weightings with a minimum number of nonnegative k-dimensional subspaces which support the stated conjecture on vector spaces.

2 Private correspondence.

BIBLIOGRAPHY

[1] A. Aguglia, A. Cossidente, and G. L. Ebert. Complete spans on Hermitian varieties. In *Proceedings of the Conference on Finite Geometries (Oberwolfach, 2001)*, volume 29, pages 7–15, 2003. (Cited on page 123.)

[2] Rudolf Ahlswede and Levon H. Khachatrian. The complete intersection theorem for systems of finite sets. *European J. Combin.*, 18(2):125–136, 1997. (Cited on page 43.)

[3] Noga Alon, Hao Huang, and Benny Sudakov. Nonnegative k-sums, fractional covers, and probability of small deviations. *J. Combin. Theory Ser. B*, 102(3):784–796, 2012. (Cited on page 44.)

[4] H. Aydinian and V. Blinovsky. A remark on the problem of nonnegative k-subset sums. *Problems of Information Transmission*, 48(4):347–351, 2012. (Cited on page 142.)

[5] John Bamberg, Jan De Beule, and Ferdinand Ihringer. Weighted tight sets and m-ovoids of finite classical polar spaces. *Preprint*, 2014. (Cited on page ix.)

[6] Linda Beukemann. *Unterstrukturen in endlichen Pollarräumen*. PhD thesis, Justus-Liebig-Universität Gießen, 2013. (Cited on page 123.)

[7] Albrecht Beutelspacher. Partial spreads in finite projective spaces and partial designs. *Math. Z.*, 145(3):211–229, 1975. (Cited on pages 46 and 47.)

[8] Thomas Bier. A distribution invariant for association schemes and strongly regular graphs. *Linear Algebra Appl.*, 57:105–113, 1984. (Cited on page 44.)

[9] Thomas Bier and Philippe Delsarte. Some bounds for the distribution numbers of an association scheme. *European J. Combin.*, 9(1):1–5, 1988. (Cited on page 44.)

[10] Thomas Bier and Nachimuthu Manickam. The first distribution invariant of the Johnson-scheme. *Southeast Asian Bull. Math.*, 11(1):61–68, 1987. (Cited on page 44.)

[11] Vladimir Blinovsky. Minimal number of edges in hypergraph guaranteeing perfect fractional matching and MMS conjecture. *ArXiv e-prints*, 2014. arXiv:1310.0989v4 [math.CO]. (Cited on page 142.)

[12] A. Blokhuis, A. E. Brouwer, A. Chowdhury, P. Frankl, T. Mussche, B. Patkós, and T. Szőnyi. A Hilton-Milner theorem for vector spaces. *Electron. J. Combin.*, 17(1):Research Paper 71, 12, 2010. (Cited on page 60.)

[13] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-regular graphs*, volume 18 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1989. (Cited on pages 3, 5, 7, 9, 17, 20, 21, 24, 26, 154, and 155.)

[14] Andries Brouwer and Joe Hemmeter. A new family of distance-regular graphs and the $\{0,1,2\}$-cliques in dual polar graphs. *European J. Combin.*, 13(2):71–79, 1992. (Cited on pages 60 and 69.)

[15] Peter J. Cameron and C. Y. Ku. Intersecting families of permutations. *European J. Combin.*, 24(7):881–890, 2003. (Cited on page 51.)

[16] Ameera Chowdhury, Ghassan Sarkis, and Shahriar Shahriari. The Manickam-Miklós-Singhi conjectures for sets and vector spaces. *J. Combin. Theory Ser. A*, 2014. To appear. (Cited on pages 45, 47, 126, 127, 130, 131, 137, and 138.)

[17] Miroslava Cimráková and Veerle Fack. Searching for maximal partial ovoids and spreads in generalized quadrangles. *Bull. Belg. Math. Soc. Simon Stevin*, 12(5):697–705, 2005. (Cited on pages 122 and 123.)

[18] Kris Coolsaet and Hendrik Van Maldeghem. Some new upper bounds for the size of partial ovoids in slim generalized poly-

gons and generalized hexagons of order (s, s^3). *J. Algebraic Combin.*, 12(2):107–113, 2000. (Cited on pages 123 and 154.)

[19] J. De Beule, A. Klein, and K. Metsch. Substructures of finite classical polar spaces. In J. De Beule and L. Storme, editors, *Current research topics in Galois geometry*, chapter 2, pages 35–61. Nova Sci. Publ., New York, 2012. (Cited on page 50.)

[20] J. De Beule, A. Klein, K. Metsch, and L. Storme. Partial ovoids and partial spreads in Hermitian polar spaces. *Des. Codes Cryptogr.*, 47(1-3):21–34, 2008. (Cited on pages 120 and 121.)

[21] M. De Boeck. The largest Erdős-Ko-Rado sets of planes in finite projective and finite classical polar spaces. *Des. Codes Cryptogr.*, Accepted(Special issue "Finite Geometries, in honor of F. De Clerck"), 2013. (Cited on pages 51 and 60.)

[22] M. De Boeck. The largest Erdős-Ko-Rado sets in $2-(v,k,1)$ designs. *Des. Codes Cryptogr.*, Accepted:doi:10.1007/s10623–014–9929–5, 2014. (Cited on page 51.)

[23] M. De Boeck. The second largest Erdős-Ko-Rado sets of generators of the hyperbolic quadrics $\mathcal{Q}^+(4n+1,q)$. *Adv. Geom.*, Submitted, 2014. (Cited on pages 51 and 60.)

[24] Bart De Bruyn and Frédéric Vanhove. Inequalities for regular near polygons, with applications to m-ovoids. *European J. Combin.*, 34(2):522–538, 2013. (Cited on page 123.)

[25] Philippe Delsarte. An algebraic approach to the association schemes of coding theory. *Philips Res. Rep. Suppl.*, (10):vi+97, 1973. (Cited on pages 17 and 18.)

[26] Philippe Delsarte. Association schemes and t-designs in regular semilattices. *J. Combinatorial Theory Ser. A*, 20(2):230–243, 1976. (Cited on page 21.)

[27] Irit Dinur and Ehud Friedgut. Intersecting families are essentially contained in juntas. *Combin. Probab. Comput.*, 18(1-2):107–122, 2009. (Cited on page 45.)

[28] R. H. Dye. Maximal sets of nonpolar points of quadrics and symplectic polarities over $GF(2)$. *Geom. Dedicata*, 44(3):281–293, 1992. (Cited on page 123.)

[29] G. L. Ebert and J. W. P. Hirschfeld. Complete systems of lines on a Hermitian surface over a finite field. *Des. Codes Cryptogr.*, 17(1-3):253–268, 1999. (Cited on page 123.)

[30] Jörg Eisfeld. The eigenspaces of the Bose-Mesner algebras of the association schemes corresponding to projective spaces and polar spaces. *Des. Codes Cryptogr.*, 17(1-3):129–150, 1999. (Cited on pages 21, 22, and 26.)

[31] David Ellis, Ehud Friedgut, and Haran Pilpel. Intersecting families of permutations. *J. Amer. Math. Soc.*, 24(3):649–682, 2011. (Cited on pages 14 and 51.)

[32] P. Erdős, Chao Ko, and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 12:313–320, 1961. (Cited on pages 42 and 43.)

[33] Gerd Fischer. *Lineare Algebra*. vieweg studium, 15 edition, 2005. (Cited on page 8.)

[34] P. Frankl. *The Erdős-Ko-Rado theorem is true for* $n = ckt$, volume 18 of *Colloq. Math. Soc. János Bolyai*. North-Holland, Amsterdam-New York, 1978. (Cited on page 43.)

[35] P. Frankl and R. M. Wilson. The Erdős-Ko-Rado theorem for vector spaces. *J. Combin. Theory Ser. A*, 43(2):228–236, 1986. (Cited on pages 22, 45, and 46.)

[36] C. D. Godsil. Graphs, groups and polytopes. In *Combinatorial mathematics (Proc. Internat. Conf. Combinatorial Theory, Australian Nat. Univ., Canberra, 1977)*, volume 686 of *Lecture Notes in Math.*, pages 157–164. Springer, Berlin, 1978. (Cited on page 18.)

[37] Chris Godsil and Karen Meagher. A new proof of the Erdős-Ko-Rado theorem for intersecting families of permutations. *European J. Combin.*, 30(2):404–414, 2009. (Cited on page 51.)

[38] Chris Godsil and Gordon Royle. *Algebraic graph theory*, volume 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001. (Cited on page 3.)

[39] Rod Gow, Michel Lavrauw, John Sheekey, and Frédéric Vanhove. Constant rank-distance sets of Hermitian matrices and partial spreads in Hermitian polar spaces. *Electron. J. Combin.*, 21(1):Paper 1.26, 19, 2014. (Cited on page 123.)

[40] Willem H. Haemers. Disconnected vertex sets and equidistant code pairs. *Electron. J. Combin.*, 4(1):Research Paper 7, 10 pp. (electronic), 1997. (Cited on page 14.)

[41] Stephen G. Hartke and Derrick Stolee. A linear programming approach to the Manickam-Miklós-Singhi conjecture. *European J. Combin.*, 36:53–70, 2014. (Cited on page 142.)

[42] A. J. W. Hilton. An intersection theorem for a collection of families of subsets of a finite set. *J. London Math. Soc. (2)*, 15(3):369–376, 1977. (Cited on page 43.)

[43] A. J. W. Hilton and E. C. Milner. Some intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 18:369–384, 1967. (Cited on page 60.)

[44] J. W. P. Hirschfeld and J. A. Thas. *General Galois geometries*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications. (Cited on pages 19, 22, and 27.)

[45] J.W.P. Hirschfeld. *Projective Geometries Over Finite Fields*. Oxford Mathematical Monographs. Clarendon Press, 1998. (Cited on pages 19 and 28.)

[46] W. N. Hsieh. Intersection theorems for systems of finite vector spaces. *Discrete Math.*, 12:1–16, 1975. (Cited on page 45.)

[47] Hao Huang and Benny Sudakov. The minimum number of nonnegative edges in hypergraphs. *Electronic Journal of Combinatorics*, 2014. To appear. (Cited on pages 47 and 126.)

[48] Ferdinand Ihringer. A characterization of the natural embedding of the split Cayley hexagon in $\mathrm{PG}(6, q)$ by intersection numbers in finite projective spaces of arbitrary dimension. *Discrete Math.*, 314:42–49, 2014. (Cited on page ix.)

[49] Ferdinand Ihringer. Cross-intersecting Erdős-Ko-Rado sets in finite classical polar spaces. *Submitted*, 2014. (Cited on page ix.)

[50] Ferdinand Ihringer. A new upper bound for constant distance codes of generators on hermitian polar spaces of type $H(2d-1, q^2)$. *J. Geom.*, 105(3):457–464, 2014. (Cited on page ix.)

[51] Ferdinand Ihringer. A note on the Manickam-Miklós-Singhi conjectures for vector spaces. *Submitted*, 2014. (Cited on page ix.)

[52] Ferdinand Ihringer and Klaus Metsch. Large $\{0, 1, \ldots, t\}$-cliques in dual polar graphs. *Preprint*, 2014. (Cited on page ix.)

[53] Ferdinand Ihringer and Klaus Metsch. On the maximum size of Erdős-Ko-Rado sets in $H(2d+1, q^2)$. *Des. Codes Cryptogr.*, 72(2):311–316, 2014. (Cited on page ix.)

[54] Petteri Kaski, Patric R. J. Östergård, Olli Pottonen, and Lasse Kiviluoto. A catalogue of the Steiner triple systems of order 19. *Bull. Inst. Combin. Appl.*, 57:35–41, 2009. (Cited on page 5.)

[55] Azadeh Khaleghi, Danilo Silva, and Frank R. Kschischang. Subspace codes. In *Cryptography and coding*, volume 5921 of *Lecture Notes in Comput. Sci.*, pages 1–21. Springer, Berlin, 2009. (Cited on page 120.)

[56] Ralf Kötter, Frank R. Kschischang, and Danilo Silva. A rank-metric approach to error control in random network coding. *IEEE Trans. Inform. Theory*, 54(9):3951–3967, 2008. (Cited on page 120.)

[57] Cheng Yeaw Ku and David Renshaw. Erdős-Ko-Rado theorems for permutations and set partitions. *J. Combin. Theory Ser. A*, 115(6):1008–1020, 2008. (Cited on page 51.)

[58] D. Luyckx. On maximal partial spreads of $H(2n+1, q^2)$. *Discrete Math.*, 308(2-3):375–379, 2008. (Cited on page 123.)

[59] Carlos J. Luz. A characterization of Delsarte's linear programming bound as a ratio bound. *Linear Algebra Appl.*, 423(1):99–108, 2007. (Cited on page 18.)

[60] N. Manickam and D. Miklós. On the number of nonnegative partial sums of a nonnegative sum. In *Combinatorics (Eger, 1987)*, volume 52 of *Colloq. Math. Soc. János Bolyai*, pages 385–392. North-Holland, Amsterdam, 1988. (Cited on page 44.)

[61] N. Manickam and N. M. Singhi. First distribution invariants and EKR theorems. *J. Combin. Theory Ser. A*, 48(1):91–103, 1988. (Cited on pages 44, 47, 49, 138, and 142.)

[62] Makoto Matsumoto and Norihide Tokushige. The exact bound in the Erdős-Ko-Rado theorem for cross-intersecting families. *J. Combin. Theory Ser. A*, 52(1):90–97, 1989. (Cited on page 43.)

[63] M. W. Newman. *Independent Sets and Eigenspaces*. PhD thesis, University of Waterloo, Waterloo, Canada, 2004. (Cited on page 45.)

[64] C.H. Papadimitriou and K. Steiglitz. *Combinatorial optimization: algorithms and complexity*. Prentice Hall, 1982. (Cited on page 18.)

[65] Stanley E. Payne and Joseph A. Thas. *Finite generalized quadrangles*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, second edition, 2009. (Cited on pages 111 and 154.)

[66] Valentina Pepe, Leo Storme, and Frédéric Vanhove. Theorems of Erdős-Ko-Rado type in polar spaces. *J. Combin. Theory Ser. A*, 118(4):1291–1312, 2011. (Cited on pages 39, 48, 57, 80, 85, 97, 98, 100, 101, 102, 103, 111, and 115.)

[67] Alexey Pokrovskiy. A linear bound on the Manickam-Miklos-Singhi conjecture. *J. Combin. Theory Ser. A*, 2014. To appear. (Cited on page 45.)

[68] Dennis Stanton. Some Erdős-Ko-Rado theorems for Chevalley groups. *SIAM J. Algebraic Discrete Methods*, 1(2):160–163, 1980. (Cited on pages 48, 54, and 97.)

[69] Dennis Stanton. Some q-Krawtchouk polynomials on Chevalley groups. *Amer. J. Math.*, 102(4):625–662, 1980. (Cited on page 158.)

[70] Dennis Stanton. A partially ordered set and q-Krawtchouk polynomials. *J. Combin. Theory Ser. A*, 30(3):276–284, 1981. (Cited on page 26.)

[71] Sho Suda and Hajime Tanaka. A cross-intersection theorem for vector spaces based on semidefinite programming. *Bull. Lond. Math. Soc.*, 46(2):342–348, 2014. (Cited on page 46.)

[72] Hajime Tanaka. Classification of subsets with minimal width and dual width in Grassmann, bilinear forms and dual polar graphs. *J. Combin. Theory Ser. A*, 113(5):903–910, 2006. (Cited on page 46.)

[73] J. A. Thas. Ovoids and spreads of finite classical polar spaces. *Geom. Dedicata*, 10(1-4):135–143, 1981. (Cited on page 120.)

[74] Jacques Tits. *Buildings of spherical type and finite BN-pairs*. Lecture Notes in Mathematics, Vol. 386. Springer-Verlag, Berlin, 1974. (Cited on pages 22 and 23.)

[75] Norihide Tokushige. The eigenvalue method for cross t-intersecting families. *J. Algebraic Combin.*, 38(3):653–662, 2013. (Cited on pages 14 and 46.)

[76] Flemming Topsøe. Some bounds for the logarithmic function. In *Inequality theory and applications. Vol. 4*, pages 137–151. Nova Sci. Publ., New York, 2007. (Cited on page 36.)

[77] Frédéric Vanhove. The maximum size of a partial spread in $H(4n+1, q^2)$ is $q^{2n+1}+1$. *Electron. J. Combin.*, 16(1):Note 13, 6, 2009. (Cited on page 123.)

[78] Frédéric Vanhove. *Incidence geometry from an algebraic graph theory point of view*. PhD thesis, University Of Ghent, 2011. (Cited on pages 21, 22, 26, 120, 122, and 123.)

[79] F. D. Veldkamp. Polar geometry. I, II, III, IV, V. *Nederl. Akad. Wetensch. Proc. Ser. A 62; 63 = Indag. Math. 21 (1959), 512-551*, 22:207–212, 1959. (Cited on page 22.)

[80] Richard M. Wilson. The exact bound in the Erdős-Ko-Rado theorem. *Combinatorica*, 4(2-3):247–257, 1984. (Cited on page 43.)

[81] Paul-Hermann Zieschang. *An Algebraic Approach to Association Schemes*. Lecture Notes in Mathematics. Springer, 1996. (Cited on page 10.)

A

A MULTIPLICITY BOUND FOR GENERALIZED POLYGONS

In this chapter we discuss the analog of Theorem 7 for all generalized polygons with parameters (s, t). We refer to [13, Section 6.5] for the necessary definitions. Note that generalized d-gons satisfy $d \in \{4, 6, 8\}$ if $s, t > 1$ [13, Th. 6.5.1].

If $d = 4$, then according [13, Table 6.4] the multiplicity of one of the eigenvalues is $s^2(st+1)/(s+t)$. Hence, by Proposition 1.15 (with a few exceptions for particular s and t), $s^2(st+1)/(s+t)$ is an upper bound on the size of an $\{i\}$-clique in a generalized quadrangle. By [13, p. 202], linear programming yields $st+1$ as an upper bound if $i = 2$. Hence, if

$$s \leqslant \frac{\sqrt{4t+1}+1}{2},$$

then the multiplicity bound is better than the linear programming bound for $i = 2$. An easy calculation shows that this happens if $t \geqslant s(s-1)$. Note that generalized quadrangles always satisfy $t \leqslant s^2$ [13, Th. 6.5.1]. Additionally, the multiplicity $s^2(st+1)/(s+t)$ has to be an integer. Hence, this bound is only useful for generalized quadrangles with parameters $(s, s^2 - s)$ and (s, s^2) (also see [65, Result 1.2.4]).

If $d = 6$, then by [13, Table 6.4] one multiplicity is

$$\frac{s^3(1+st+s^2t^2)}{s^2+st+t^2}.$$

By [13, p. 202], the linear programming bound for a $\{2\}$-clique is

$$1 + \frac{s^2t(t+1)}{(s-1)\sqrt{st}+s}.$$

It is easy to verify that for $t = s^3$ the multiplicity bound is better. See also Coolsaet and Van Maldeghem [18], who first observed this bound and even improved it further by one. Also note that $s \leqslant t^3$ [13, Th. 6.5.1].

If $d = 8$, then by [13, Table 6.4] one multiplicity is

$$\frac{s^4(1+st)(1+s^2t^2)}{(s+t)(s^2+t^2)}.$$

By [13, p. 203], the linear programming bound for a $\{4\}$-clique is

$$s^6 + 1$$

and even better for other $\{i\}$-cliques. As we have $s \leqslant t^2$ [13, Th. 6.5.1], here the linear programming bound is always better than the multiplicity bound.

For all other multiplicities and choices of i, the multiplicity bound is worse than the linear programming bound.

B

OPEN PROBLEMS

Here is a list of open problems which I encountered during my PhD and which I consider interesting. Some are already mentioned in the individual chapters.

- Partial Spreads of $H(2d-1, q^2)$, d even. Fix d. Many people assume that partial spreads have at most size $O(q^{2d-2})$. Prove that a partial spread has at most size $O(q^{2d-1-e})$ for some $e > 0$. Alternatively, construct an infinite class of spreads of size $O(q^{2d-2+e})$.

- Is the bound given in Theorem 7.4 tight for any other choice of d, i and q besides $(d, i, q) = (2, 2, 2)$ and $(d, i, q) = (3, 2, 2)$?

- Generalized quadrangles with parameters (s, s^2) are not classified. Does there exist a quadrangle for which the bound given in Theorem 7, respectively, Chapter A is tight besides $H(3, 4)$? This question is due to Jason Williford.

- Cross-intersecting EKR sets on $H(2d-1, q^2)$, $d > 2$ odd. Are the examples given in Chapter 6 the largest?

- Cross-interesting EKR sets in general. What are tight bounds on cross-intersecting EKR sets of totally isotropic subspaces, which are not generators? This question is due to Klaus Metsch.

- EKR sets, $H(2d-1, q^2)$, $d > 3$ odd. Which size have the largest examples? What are the largest examples?

- Is Conjecture 5.40 on the maximum size of EKR sets true?

- After I published a preprint of the paper version of Chapter 6, Dennis Stanton asked me the following question by e-mail (and in a publication from 1980 [69]). Are there perfect 1-codes in $Q(2d, q)$ or $W(2d-1, q)$ if $d = 2^h - 1$ for integer h? Here a perfect 1-code is a set Y of generators such that every generator meets exactly one generator of Y in a subspace of at most codimension 1.

- Is the MMS conjecture for vector spaces true for all $n \geqslant 2k$ and q?

C

SELBSTÄNDIGKEITSERKLÄRUNG

Ich erkläre:

Ich habe die vorgelegte Dissertation selbständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt, die ich in der Dissertation angegeben habe. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben, die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht. Bei den von mir durchgeführten und in der Dissertation erwähnten Untersuchungen habe ich die Grundsätze guter wissenschaftlicher Praxis, wie sie in der „Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis" niedergelegt sind, eingehalten.

Gießen, Februar 2015

Ferdinand Ihringer