The Erdős-Rado Sunflower Problem for Vector Spaces

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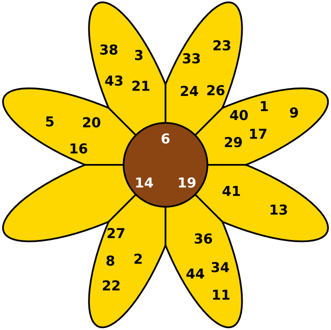
Sunflowers

An **s-sunflower**[1] is a family $\mathcal{S}$ of sets such that

1. we have $|\mathcal{S}| = s$, and
2. there exists a **kernel** K such that $S \cap T = K$ for all distinct $S, T \in \mathcal{S}$.



[1]Or Δ-system. The word **Δ-system** was used by Erdős and Rado in 1960. Frankl and Deza used the word **sunflower** exactly **once** in a paper from 1981. The Boolean Circuit Complexity community then popularized it and now it is the norm.

The Erdős-Rado Conjecture

Lemma (Erdős, Rado (1960))

Let $\mathcal{F}$ be a maximum s-sunflower-free family of k-sets. Then

$$(s-1)^k \leq |\mathcal{F}| \leq k!(s-1)^k.$$

Upper Bound: induction on k.

Lower Bound: complete k-partite hypergraph with parts of size $s-1$.

Conjecture (Erdős-Rado (1960))

For $s \geq 3$, we have $|\mathcal{F}| \leq (Cs)^k$ for some universal constant C.

Alweiss-Lovett-Wu-Zhang

Let $\mathcal{F}$ be a maximum s-sunflower-free family of k-sets.

Recall:

Conjecture (Erdős, Rado (1960))

For $s \geq 3$, we have $|\mathcal{F}| \leq (Cs)^k$ for some universal constant C.

Breakthrough:

Theorem (Alweiss, Lovett, Wu, Zhang (Annals of Mathematics, 2021); Bell, Chueluecha, Warnke (improvement, 2021))

For $s \geq 3$, we have $|\mathcal{F}| \leq (Cs \log k)^k$ for some universal constant C.

This result had countless implications. At the end we will discuss one for **intersecting families**.

Definition for Vector Spaces

What is the **right definition** for vector spaces?

Variant 1: $S \cap T$ is independent of $S, T \in \mathcal{S}$. This is one in some existing literature, see Etzion and Raviv (2015).[2]

[2] Motivated by **subspace codes**.

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A s-sunflower $\mathcal{S}$ of k-sets with kernel of size d satisfies both

1. $|\bigcap_{S \in \mathcal{S}} S| = d$, and
2. $|\bigcup_{S \in \mathcal{S}} S| = s(k-d)+d$.

Variant 2: Maybe we want all of

1. $\dim\left(\bigcap_{S \in \mathcal{S}} S\right) = d$, and
2. $\dim\left(\sum_{S \in \mathcal{S}} S\right) = s(k-d)+d$.

We will go with this.

[2] Motivated by **subspace codes**.

Examples

The (almost?) **best example for sets** generalizes to

Example

1. Fix k subspaces of dimension $s-1$ in general position.
2. Take all k-spaces which meet all of these.

Size: $[s-1]^k \geq q^{(s-1)k}$.

Better:

Example

1. Fix a vector space of dimension $\frac{(s+1)k}{2}-1$.
2. Take the largest number of k-spaces such that no $(\frac{k}{2}+1)$-space lies in more than one.
3. This uses **subspace codes**.

Size: $\geq q^{\frac{(s-1)k^2}{4}+\frac{(s-2)k}{2}-1}$.

Main Theorem

Our problem is **distinct** from the problem for sets:[3]

Theorem (Ih., Kupavskii (2025*, accepted in FFA))

*Let $s \geq 3$, $k \geq 2$, and q a prime power. Let $\mathcal{F}$ be an **s-sunflower-free family of k-spaces** over the finite field with q elements. Then*

$$|\mathcal{F}| \leq \prod_{i=1}^{k} [i(s-1)]_q \leq \left(\frac{q}{q-1}\right)^k q^{(s-1)\binom{k+1}{2}-k}.$$

*Moreover, there exist such families $\mathcal{F}$ that satisfy the following **bounds**:*

1. *For $s \geq k+1$,*
$$|\mathcal{F}| \geq q^{(s-1)\binom{k+1}{2}-k}.$$
2. *For $s \leq k$,*
$$|\mathcal{F}| \geq q^{(s-2)\binom{k+1}{2}+k\frac{\gcd(k,s-1)-3}{2}} \geq q^{(s-2)\binom{k+1}{2}-k}.$$

[3] Which makes it at least a little bit **interesting**.

The Upper Bound

Recall **easy bound** for sets: $k!(s-1)^k$.

Lemma

Let $s \geq 3$. Let $\mathcal{F}$ be an s-sunflower-free family of k-spaces. Then $|\mathcal{F}| \leq [k(s-1)] \cdot [(k-1)(s-1)] \cdots [s-1]$.

Proof.

Induction on k. □

Here $[m] := \frac{q^m-1}{q-1}$.

This easy proof **also motivates** our definition (Variant 2).

The Lower Bound

We claim the following constructions:

1. For $s \geq k+1$,
$$|\mathcal{F}| \geq q^{(s-1)\binom{k+1}{2}-k}.$$
2. For $s \leq k$,
$$|\mathcal{F}| \geq q^{(s-2)\binom{k+1}{2}+k\frac{\gcd(k,s-1)-3}{2}} \geq q^{(s-2)\binom{k+1}{2}-k}.$$

They use large **maximum rank distance (MRD) codes**.

Toy Example 1

Recall:

$$|\mathcal{F}| \leq (1 + \tfrac{1}{q-1})^k q^{(s-1)\binom{k+2}{2}-k}.$$

Example

1. Fix a vector space of dimension $\frac{(s+1)k}{2} - 1$.
2. Take the largest number of k-spaces such that no $(\frac{k}{2}+1)$-space lies in more than one.
3. This uses **subspace codes**.

Size: $\geq q^{\frac{(s-1)k^2}{4} + \frac{(s-2)k}{2} - 1}$.

Can we close the gap?

Idea: Recursion.

Toy Example 2

Consider $s = 3$ and $k = 2$.

Work in $V(5, q)$.

1. Fix a 1-space T.
2. Through T we find $q^2 + 1$ 3-spaces $\Pi_1, \Pi_2, \ldots, \Pi_{q^2+1}$ whose pairwise intersection is T.[4]
3. In Π_1, pick all $q^2 + q + 1$ 2-spaces.
4. In Π_i, $i > 1$, pick all q^2 2-spaces disjoint from T.

Size: $q^4 + q^2 + q + 1$.

Upper Bound: $(q^3 + q^2 + q + 1)(q + 1)$.

[4] If you are a finite geometer, then you will know that this is a **spread** of $PG(3, q)$, that is, a partition of $V(4, q) \cong V(5, q)/T$ into 2-spaces.

Gabidulin Codes

For our construction, we need a family of subspaces $\mathcal{C}$ such that:

1. No d-space is in more than one element of $\mathcal{C}$.
2. All elements of $\mathcal{C}$ are disjoint from a given subspace T.

This is precisely an **MRD code**.

Theorem (Gabidulin (1985))

For any $n \geq m \geq d \geq 0$, there exists a family of m-spaces $\mathcal{C}$ in $V(n,q)$ of size $\mathbf{q^{d(n-m)}}$ if $n \geq 2m$, respectively, of size $\mathbf{q^{m(n-2m+d)}}$ if $n \leq 2m$ and $n-2m+d \geq 0$, such that

1. *any **$\mathbf{d}$-space lies in at most one element** of $\mathcal{C}$,*
2. ***all** elements of $\mathcal{C}$ are **disjoint from a fixed $\mathbf{(n-m)}$-space**.*

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$$|\mathcal{F}| \leq \prod_{i=1}^{k} [i(s-1)]_q \leq \left(\frac{q}{q-1}\right)^k q^{(s-1)\binom{k+1}{2}-k}.$$

*Moreover, there exist such families $\mathcal{F}$ that satisfy the following **bounds**:*

1. *For $s \geq k+1$,*
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Future Work

For $s \geq k+1$, **our result is complete** up to a constant.

Question

Is our construction also (asymptotically) optimal for $s \leq k$?

Maybe here is still some room for the **Alweiss-Lovett-Wu-Zhang argument**.

Question

What is the precise upper bound for small parameters?

This sounds like a good problem for **coding theorists** and **finite geometers**.

Question

Is our construction for $s \geq k+1$ always in a maximum example?

For $s \leq k$, this seems to be false (at least in an asymptotic sense). **There is choice** on how to pick a construction there and we only described one possible option.

Application to Intersecting Families

Theorem (Alweiss, Lovett, Wu, Zhang (Annals of Mathematics, 2021); Bell, Chueluecha, Warnke (improvement, 2021))

For $s \geq 3$, we have $|\mathcal{F}| \leq (Cs \log k)^k$ for some universal constant C.

As a consequence: the **spread approximation technique** for intersection problems by Kupavskii and Zakharov (Advances in Mathematics, 2024).

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Makes many EKR-problems much easier, for instance:

Theorem (Frankl, Hurlbert, Ih., Kupavskii, Lindzey, Meagher, Pantangi (2025*))

For $n \geq 2^{19}$, let $\mathcal{F}$ be a 1-intersecting family of trees on n vertices. Then

$$|\mathcal{F}| \leq 2n^{n-3} + (n-2).$$

Moreover, equality holds if and only if $\mathcal{F}$ consists of all stars plus all trees containing one fixed edge.

EKR Methods: Sets versus Vector Spaces

Our result strongly suggests that the **spead approximation technique** is **not useful in vector spaces** and similar settings.

Thus, we have the following **picture of EKR techniques**, graded by Chinese university grades:

Method	$\{1, \ldots, n\}$	$V(n, q)$
Kernel	70	85
Shifting	100	0
Katona Cycle	75	40
Eigenvalues	90	90
Rank Arguments	85	85
Junta	90	60(?)
Spread	90	**10**
Onion Peeling	75	**90(?)**

Table: My Ranking of EKR-Methods.

Onion peeling for vector spaces is **ongoing work** with Andrey.

Thank you for your attention!