Erdős-Ko-Rado Sets in Finite Classical Polar Spaces

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Erdős-Ko-Rado Sets of Sets

Definition

Let $n \geq 2k$. Consider $X = \{1, \dots, n\}$. An **Erdős-Ko-Rado set** (EKR set) of X is a set Y of k-subsets of X such that the elements of Y are pairwise not disjoint.

Example

1. All k-sets that contain 1. For $n = 4$, $k = 2$:

$$\{1,2\}, \{1,3\}, \{1,4\}.$$

2. $n = 2k$: All k-sets that do not contain n. For $n = 4$, $k = 2$:

$$\{1,2\}, \{1,3\}, \{2,3\}.$$

Erdős-Ko-Rado Sets of Sets

Theorem (Erdős, Ko, Rado (1961))

Let $n \geq 2k$. Consider $X = \{1, \ldots, n\}$. Let Y be a EKR set. Then

$$|Y| \leq \binom{n-1}{k-1}.$$

with equality if and only if

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Maximal examples

Definition

An EKR set Y is called **maximal** if for all k-sets $y \subseteq X$ either $y \in Y$ or $Y \cup \{y\}$ is not an EKR set.

Example

1. For $n = 4$, $k = 2$:

$$\{1,2\}, \{1,3\}, \{1,4\} \text{ is maximal.}$$

2. For $n = 4$, $k = 2$:

$$\{1,2\}, \{1,3\} \text{ is not maximal.}$$

Problem

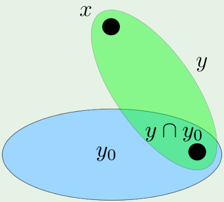
Classification of all maximal EKR sets.

The second largest maximal Examples

Example (Hilton-Milner (1967))

1. Let y_0 be a k-set and $x \in X \setminus y_0$. Then

$$Y = \{y_0\} \cup \{y \ k\text{-subset} \mid y \cap y_0 \neq \emptyset \text{ and } x \in y\}$$



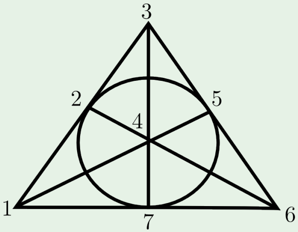
2. For $k = 3$: All 3-sets that meet a fixed 3-set in at least 2 elements.

A very small maximal example

Example

For $n \geq 7$, $k = 3$, this example is maximal:

$$\{1,2,3\}, \{1,4,5\}, \{1,6,7\},$$
$$\{2,4,6\}, \{2,5,7\}, \{3,4,7\}, \{3,5,6\}.$$



This is PG(2, 2). No intersections in 2-sets.
All similar examples were investigated by Eisfeld (1999).

Erdős-Ko-Rado Sets in Projective Spaces

Definition

An **Erdős-Ko-Rado set** Y of k-spaces of a projective space $\mathrm{PG}(n, q)$ is a set of k-spaces that are pairwise not disjoint.

Erdős-Ko-Rado Sets in Projective Spaces

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An **Erdős-Ko-Rado set** Y of k-spaces of a projective space PG(n, q) is a set of k-spaces that are pairwise not disjoint.

Theorem (Hsieh (1975), Frankl and Wilson (1986), N.N.)

Let $n \geq 2k$. Let Y be a EKR set of k-spaces of PG(n, q). Then

$$|Y| \leq \begin{bmatrix} n \\ k \end{bmatrix}.$$

with equality if and only if

1. *Y is the set of all k-spaces through a fixed point, or*
2. *$n = 2k + 1$ and Y is the set of all k-spaces contained in a fixed $(n-1)$-space.*

The same examples as in the original problem!

Some Notable Results for PG(n, q)

1. Hsieh (1975): Proof of upper bounds and classification of examples of maximum size for many n and k.

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5. Blockhuis, Brouwer, Chowdhury, Frankl, Mussche, Patkós, Szőnyi (2010): Classification of second largest maximal examples.

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2. All generators that meet a fixed plane in at least a line.

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1. All generators through a fixed point.
2. All generators that meet a fixed plane in at least a line.
3. All generators that meet a fixed 4-space in at least a plane.
4. If d even: All generators that meet a fixed d-space in at least a $d/2$-space.

Some History

History

1. Stanton (1980): Upper bounds for finite classical polar spaces.
2. Pepe, Storme, Vanhove (2011): Classification of all EKR sets of maximum size. Exception: $H(2d+1, q^2)$, $d > 2$ even.
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For all classical finite polar spaces, the EKR set of all generators through a fixed point has maximum size with the following exceptions:

1. $Q^+(2d+1, q)$, *d even: all generators of one type (latins/greeks) are the only maximal example.*
2. $H(2d+1, q^2)$, *d even: only for $d = 2$ the largest example is known.*

Examples in $H(2d+1, q^2)$

Example ($H(5, q^2)$)

1. All generators through a fixed point: $|Y| \approx q^4$.
2. All generators through the lines of a fixed plane: $|Y| \approx q^5$.

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Example ($H(9, q^2)$)

1. All generators through a fixed point: $|Y| \approx q^{16}$.
2. All generators through the planes of a fixed generator: $|Y| \approx q^{16}$.

Example ($H(2d+1, q^2)$)

1. All generators through a fixed point: $|Y| \approx q^{d^2}$.
2. All generators through the $d/2$-dimensional subspaces of a fixed generator: $|Y| \approx q^{\frac{3}{4}d^2+d}$.

A New Result

Theorem (Stanton (1980))

If $H(2d+1, q^2)$, $d > 2$ even, then

$$|Y| \lessapprox q^{d^2+d}.$$

Theorem (Ihringer, Metsch (2012))

If $H(2d+1, q^2)$, $d > 2$ even, then

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The Hoffman Bound

Stanton's Approach

Y is a **coclique** of the **disjointness graph** Γ of the set of generators.

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Let x, y be generators. The $n \times n$ adjacency matrix A_d of Γ defined by

$$(A_d)_{xy} := \begin{cases} 1 & \text{if } x \text{ and } y \text{ disjoint,} \\ 0 & \text{otherwise.} \end{cases}$$

has $d + 2$ eigenvalues $\lambda_{-1}, \lambda_0, \ldots, \lambda_d$.

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Theorem (Hoffman Bound)

Let k be the valency of Γ and $\lambda_{\min} < 0$ the smallest eigenvalue of A_d, then

$$|Y| \leq \frac{n\lambda_{\min}}{\lambda_{\min} - k} \approx q^{d^2+d}.$$

There exist symmetric matrices $E_{-1}, \dots, E_d$ with

$$\chi^t E_i \chi \geq 0$$

and some other useful properties. E.g.

$$I = \sum_{i=-1}^{d} E_i, \qquad J = nE_{-1}, \qquad A_d = \sum_{i=-1}^{d} \lambda_i E_i.$$

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Let χ be the characteristic vector of Y. Then $\chi^t A_d \chi = 0$.

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Idea Hoffman Bound.

Let χ be the characteristic vector of Y. Then $\chi^t A_d \chi = 0$. Calculating

$$0 \leq n\chi^t \left(\sum_{i=0}^{d} (\lambda_i - \lambda_{\min}) E_i \right) \chi$$

yields the Hoffman bound

$$|Y| \leq \frac{n\lambda_{\min}}{\lambda_{\min} - k} \approx q^{d^2+d}.$$

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Calculation: Hoffman Bound

Calculating

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Calculation: New Result

Calculating

$$0 \leq n\chi^t \left(\sum_{i=1}^{d} (\lambda_i - \lambda_{\min}) E_i \right) \chi$$

yields the main result

$$|Y| \lessapprox q^{d^2+1}.$$

Definition

The **inner distribution vector** of Y is $y = (1, y_0, \dots, y_{d-1}, y_d)$, where y_i is the average number of intersection in dimension $d - i - 1$:

$$y_i := |\{(a, b) \in Y^2 \mid \dim(a \cap b) = d - i - 1\}|/|Y|.$$

By definition $y_d = 0$, since Y is an EKR set.

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By definition $y_d = 0$, since Y is an EKR set.

Theorem (Delsarte's Linear Programming Bound)

There exists a matrix Q such that

$$(yQ)_i = \chi^t E_i \chi \geq 0.$$

Hence this is a linear optimization problem!

Q for generators of $H(5, q^2)$

$$0 \leq yQ \approx (1, y_0, y_1, 0) \begin{pmatrix} 1 & q^8 & q^9 & q^5 \\ 1 & q^6 & -q^6 & -q^4 \\ 1 & q^4 & -q^4 & q^3 \\ 1 & -q^3 & q^3 & -q^2 \end{pmatrix}$$

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Hoffman Bound (here: $\lambda_{\min} = \lambda_2 < \lambda_0$):

$$0 \leq n \sum_{i=0}^{2} (\lambda_i - \lambda_{\min})(yQ)_i.$$

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The Hoffman bound adds the equation

$$0 \leq q^8 + q^6 y_0 + q^4 y_1$$

to the sum (multiplied by some positive factor), since λ_0 is not the smallest eigenvalue. But: $y_i \geq 0$. This equation only worsens the result.

Limits of the Approach

Hoffman bound and new result consider only special solutions of the linear optimization problem.

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$H(5, q^2)$

1. Hoffman bound: $|Y| \leq q^6 + q^5 + q + 1$.
2. New result: $|Y| \leq q^5 + q^4 + q^3 + 1$.
3. Linear programming bound: $|Y| \leq q^5 + q^4 + q^3 + 1$.
4. Largest example (Pepe, Storme, Vanhove): $|Y| = q^5 + q^3 + q + 1$.

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4. Largest example (Pepe, Storme, Vanhove): $|Y| = q^5 + q^3 + q + 1$.

$H(9, q^2)$

1. New result: $|Y| \leq q^{17} + 3q^{16} + 4q^{15} + 5q^{14} + \ldots$.
2. Linear programming bound: $|Y| \leq q^{17} + 2q^{16} + 2q^{15} + q^{14} + \ldots$.
3. Largest known example: $|Y| \approx q^{16}$.

Known Limits

New result

$$|Y| \lesssim q^{d^2+1}.$$

Largest known example for $d > 2$ even

$$|Y| \approx q^{d^2}.$$

Here Y is the set of all generators through a fixed point.

Known Limits

New result

$$|Y| \lessapprox q^{d^2+1}.$$

Largest known example for $d > 2$ even

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Here Y is the set of all generators through a fixed point.

Conjecture

The largest known example is the largest example.

Question

Is there an generalization of the largest example of $H(5, q^2)$ to a largest example of $H(2d + 1, q^2)$?

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Answer

Yes.

Definition

An $(d, d-t)$-**Erdős-Ko-Rado set** Y of generators (maximal totally isotropic subspaces) of a polar space is a set of generators that intersect pairwise in at least a $(d-t)$-space.

$(d, d-2)$ EKR sets in $H(2d+1, q^2)$

Example ($(2,0)$ EKR sets of $H(5, q^2)$)

1. All generators through a fixed point: $|Y| = |\Sigma H(3, q^2)| \approx q^4$.
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Example ((3, 1) EKR sets of $H(7, q^2)$)

1. All generators through a fixed line: $|Y| = |\Sigma H(3, q^2)| \approx q^4$.
2. All generators through the planes of a fixed 3-space: $|Y| \approx q^7$

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Example ($(d, d-2)$ EKR sets of $H(2d+1, q^2)$)

1. All generators through a fixed $(d-2)$-space: $|Y| \approx q^4$.
2. All generators through the $(d-1)$-dimensional subspaces of a fixed generator: $|Y| \approx q^{2d+1}$.

$(d, d-2)$ EKR set of $Q^-(2d+3, q)$

Example ((2, 0) EKR sets of $Q^-(7, q)$)

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Main result on $(d, d-t)$ EKR Sets

First Thoughts

For large d, the largest $(d, d-2)$ EKR set is the set of all generators meeting a fixed generator in at least a $(d-1)$-space.

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Indeed, a very general result is possible:

Theorem (Ihringer, Metsch (2013))

In all classical finite polar spaces for $3t \lessapprox 2d$:

1. *t even: The largest $(d, d-t)$ EKR set is the set of all generators meeting a fixed d-space in at least a $(d - \frac{t}{2})$-space.*

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2. *t odd: The largest $(d, d-t)$ EKR set is the set of all generators meeting a fixed $(d-1)$-space in at least a $(d-\frac{t}{2}-\frac{1}{2})$-space.*

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2. *t odd: The largest $(d, d-t)$ EKR set is the set of all generators meeting a fixed $(d-1)$-space in at least a $(d - \frac{t}{2} - \frac{1}{2})$-space.*

Particularly, this implies a complete classification of $(d, d-2)$ EKR sets and a nearly complete classification of $(d, d-3)$ EKR sets.

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2. Show that Y is small, not maximal, or the desired example.

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1. The number of generators of the polar space.
2. Calculate smallest eigenvalue and apply Hoffman bound.
3. Apply the $H(2d+1, q^2)$, d even, trick.
4. Delsarte's linear programming bound.
5. Correct bound (i.e. known for $d = t$ in the most cases).

Part 2: Y is small, not maximal, or the desired example.

Sketch of the proof for $d = 3, t = 2$: $(3, 1)$ EKR sets.

What do we want to show?
If Y is large, then there exists a 3-space/generator U such that all elements of Y meet U in at least a plane.

Part 2: Y is small, not maximal, or the desired example.

Sketch of the proof for $d = 3, t = 2$: $(3, 1)$ EKR sets.

What do we want to show?
If Y is large, then there exists a 3-space/generator U such that all elements of Y meet U in at least a plane.
So we have to ...

1. ... find an appropriate 3-space U.
2. ... show that all elements of Y meet U in at least a plane.

□

Sketch of the proof for $d = 3, t = 2$.

Three generators of Y have a common point for dimensional reasons.

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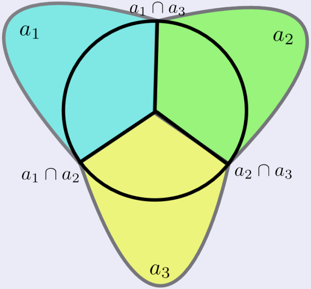

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Hence $U := \langle a_1 \cap a_2, a_1 \cap a_3, a_2 \cap a_3 \rangle$ is a generator.

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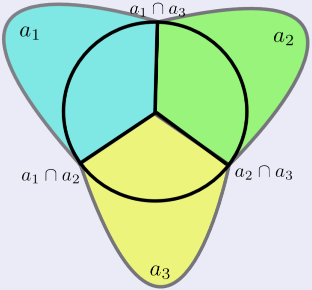

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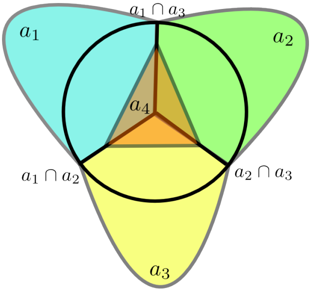
Case 1: All generators contain P: Y/P is an $(2, 0)$ EKR set of $P^\perp/P$. □

Case 2: Not all elements of Y contain P. Consider a a_4 s.t. $P \notin a_4$:



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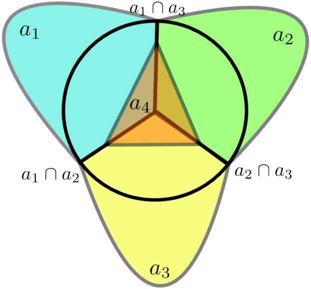
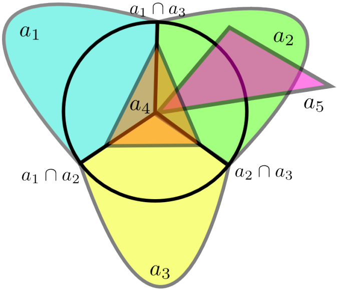


Then a_4 meets $a_1 \cap a_2$, $a_1 \cap a_3$, and $a_2 \cap a_3$ in a point. Hence a_4 meets $U = \langle a_1 \cap a_2, a_1 \cap a_3, a_2 \cap a_3 \rangle$ in at least a plane. Good!

Assume that there exists a generator $a_5 \in Y$ s.t. $a_5 \cap U$ is at most a line.



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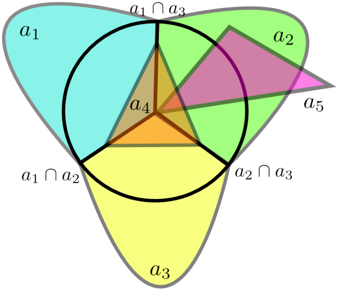

a_5 behaves to a_1, a_2, a_4 as a_4 to a_1, a_2, a_3. Hence a_5 has to meet U in a plane! Also good! □

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4. Stability theorems for small d.