Erdős-Ko-Rado Sets in Hermitian Polar Spaces

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Erdős-Ko-Rado Sets

Theorem (Erdős, Ko, Rado (1961))

Let $n \geq 2k$. Consider $X = \{1, \ldots, n\}$. Let Y be a set of k-subsets of X such that the elements of Y are pairwise not disjoint. Then

$$|Y| \leq \binom{n-1}{k-1}.$$

Example

1. All k-sets that contain 1. For $n = 4$, $k = 2$:

$$\{1, 2\}, \{1, 3\}, \{1, 4\}.$$

2. $n = 2k$: All k-sets that do not contain 1. For $n = 4$, $k = 2$:

$$\{2, 3\}, \{2, 4\}, \{3, 4\}.$$

Erdős-Ko-Rado Sets in Polar Spaces

Definition

An **Erdős-Ko-Rado set** Y of generators of a polar space is a set of generators that are pairwise not disjoint.

Example

All generators through a fixed point.

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History

1. Stanton (1980): Upper bounds for finite classical polar spaces.
2. Pepe, Storme, Vanhove (2011): Classification of examples of maximum size.
3. Exception: $H(2d+1, q^2)$, $d > 2$ even.

Examples in $H(2d+1, q^2)$

Example ($H(5, q^2)$)

1. All generators through a fixed point: $|Y| \approx q^4$.
2. All generators through the lines of a fixed plane: $|Y| \approx q^5$.

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Example ($H(9, q^2)$)

1. All generators through a fixed point: $|Y| \approx q^{16}$.
2. All generators through the planes of a fixed generator: $|Y| \approx q^{16}$.

Example ($H(2d+1, q^2)$)

1. All generators through a fixed point: $|Y| \approx q^{d^2}$.
2. All generators through the $d/2$-dimensional subspaces of a fixed generator: $|Y| \approx q^{\frac{3}{4}d^2+d}$.

The Main Result

Theorem (Stanton (1980))

If $H(2d+1, q^2)$, $d > 2$ *even, then*

$$|Y| \lessapprox q^{d^2+d}.$$

Theorem (Ihringer, Metsch)

If $H(2d+1, q^2)$, $d > 2$ *even, then*

$$|Y| \lessapprox q^{d^2+1}$$

The Hoffman Bound

Stanton's Approach

Y is a coclique of the disjointness graph Γ of the set of generators.

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Let x, y be generators. The adjacency matrix A_d of Γ defined by

$$(A_d)_{xy} := \begin{cases} 1 & \text{if } x \text{ and } y \text{ disjoint,} \\ 0 & \text{otherwise.} \end{cases}$$

has $d + 2$ eigenvalues $\lambda_{-1}, \lambda_0, \ldots, \lambda_d$.

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Theorem (Hoffman Bound)

Let k be the valency of Γ and $\lambda_{\min} < 0$ the smallest eigenvalue of A_d, then

$$|Y| \leq \frac{n\lambda_{\min}}{\lambda_{\min} - k} \approx q^{d^2+d}.$$

There exist symmetric matrices $E_{-1}, \ldots, E_d$ with

$$\chi^t E_i \chi \geq 0$$

and some other useful properties. E.g.

$$I = \sum_{i=-1}^{d} E_i, \qquad J = nE_{-1}, \qquad A_d = \sum_{i=-1}^{d} \lambda_i E_i.$$

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Idea Hoffman Bound.

Let χ be the characteristic vector of Y. Then $\chi^t A_d \chi = 0$.

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Idea Hoffman Bound.

Let χ be the characteristic vector of Y. Then $\chi^t A_d \chi = 0$. Calculating

$$0 \leq n\chi^t \left(\sum_{i=0}^{d} (\lambda_i - \lambda_{\min}) E_i \right) \chi$$

yields the Hoffman bound

$$|Y| \leq \frac{n\lambda_{\min}}{\lambda_{\min} - k} \approx q^{d^2+d}.$$

□

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Idea Main Result

Calculating

$$0 \leq n\chi^t \left(\sum_{i=1}^{d} (\lambda_i - \lambda_{\min}) E_i \right) \chi$$

yields the main result

$$|Y| \lessapprox q^{d^2+1}.$$

Known Limits

Main Result

$$|Y| \lessapprox q^{d^2+1}.$$

Largest known example for $d > 2$ even

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Here Y is the set of all generators through a fixed point.

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Conjecture

The largest known example is the largest example.

Limits of the Approach

In truth $\chi^t E_i \chi \geq 0$ are linear inequalities (Delsarte).
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$H(5, q^2)$

1. Main Result: $|Y| \leq q^5 + q^4 + q^3 + 1$.
2. Linear programming bound: $|Y| \leq q^5 + q^4 + q^3 + 1$.
3. Largest example (Pepe, Storme, Vanhove): $|Y| = q^5 + q^3 + q + 1$.

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$H(9, q^2)$

1. Main Result: $|Y| \leq q^{17} + 3q^{16} + 4q^{15} + 5q^{14} + \ldots$.
2. Linear programming bound: $|Y| \leq q^{17} + 2q^{16} + 2q^{15} + q^{14} + \ldots$.
3. Largest known example: $|Y| \approx q^{16}$.