Intriguing Sets in Quadrangles and Hexagons

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Table of Contents

Generalized Quadrangles

Definition

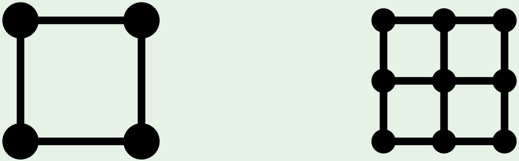
Let $\mathcal{P}$ (points) and $\mathcal{L}$ (lines) be disjoint sets. Let $\mathbf{I} \subseteq \mathcal{P} \times \mathcal{L}$ an **incidence relation**. The structure $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a generalized quadrangle of order (s, t) if and only if

(I) Each line contains exactly $s + 1$ points.

(II) Each point lies in exactly $t + 1$ lines.

(III) For every point p not on a line ℓ, there exists a unique point $p' \mathbf{I} \ell$ and a unique line $\ell' \mathbf{I} p$ such that $p' \mathbf{I} \ell'$.

Example (Order $(1, 1)$ and $(2, 1)$)



Quadrangles as SRGs

Definition

A **strongly regular graph** (SRG) with parameter (v, k, λ, μ) is a k-regular graph on v vertices s.t. two adjacent vertices have λ common neighbours and two non-adjacent vertices have μ common neighbours.

Quadrangles as SRGs

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Definition

The **point graph** of a generalized quadrangle $(\mathcal{P}, \mathcal{L}, \mathbf{I})$ is the graph with vertex set $\mathcal{P}$ s.t. two elements of $\mathcal{P}$ are adjacent iff they are in one common line.

Lemma

The point graph of a generalized quadrangle of order (s, t) is a SRG with

$$v = (s+1)(st+1) \qquad k = (t+1)s$$
$$\lambda = s-1 \qquad \mu = t+1.$$

Properties of SRGs

Definition

The **adjacency matrix** $A \in \mathbb{C}^{v \times v}$ of the graph on v vertices:

$$(A)_{xy} = \begin{cases} 1 & \text{if } x \sim y \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency matrix of a SRG (v, k, λ, μ) has only three eigenvalues k, e^+ and e^- with corresponding eigenspaces $\langle \mathbf{j} \rangle, V^+, V^-$.
Hoffman's bound:

- A set $\mathcal{T}$ of pairwise adjacent vertices (**clique**) in an SRG has at most size $1 - k/e^-$. Equality: $\chi_{\mathcal{T}} \in \langle \mathbf{j} \rangle \perp V^+$.
- A set of pairwise non-adjacent vertices (**cocliques**) k-regular graph has at most size $\frac{ve^-}{e^- - k}$. Equality: $\chi_{\mathcal{O}} \in \langle \mathbf{j} \rangle \perp V^-$.

Example

A quadrangle of order $(2, 1)$ has 9 points and this adjacency matrix:

$$\begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0
\end{pmatrix}$$

Eigenvalues: 4, 1, -2.

- Bound cliques: $1 - k/e^- = 1 - 4/2 = 3$ (lines!).
- Bound cocliques: $\frac{v \cdot e^-}{e^- - k} = \frac{9 \cdot (-2)}{-2-4} = 3$ (ovoids!).

Picture White-/Blackboard.

Partial Ovoids

Definition (Thas (1981))

A **(partial) ovoid** of a GQ is a point set, which meets every lines (at most) exactly once.

Definition (Alternative)

A **partial ovoid** of a GQ is a coclique of its point graph.

Lemma

A (partial) ovoid $\mathcal{O}$ of a GQ of order (s, t) has (at most) size $st + 1$.

Proof.

Version 1. Double counting: each of the $(t + 1)(st + 1)$ lines contains at most 1 element of $\mathcal{O}$. Each element of $\mathcal{O}$ lies on $t + 1$ lines.

Version 2. Hoffman's eigenvalue bounds. □

Existence

Lemma (Duality)

If $(\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a generalized quadrangle of order (s, t), then $(\mathcal{L}, \mathcal{P}, \mathbf{I})$ is a generalized quadrangle of order (t, s).

For a generalized quadrangles of order (s, t) with $1 < s \leq t$ only the following are known to exist: (q, q), (q, q^2), (q^2, q^3), $(q - 1, q + 1)$.

Here q is a prime power.

Example ($Q(4, q)$)

Let $Q : \mathbb{F}_q^5 \longrightarrow \mathbb{F}_q$ with $Q(x) = x_1^2 + x_2x_3 + x_4x_5$.

- Let $\mathcal{P}$ be the set of all 1-spaces that vanish on Q.
- Let $\mathcal{L}$ be the set of all 2-spaces that vanish on Q.
- Let $\mathbf{I}$ be induced by $\mathbb{F}_q^5$.

Then $(\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a generalized quadrangle of order (q, q): $Q(4, q)$.

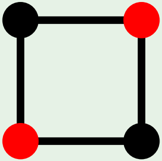
Motivation and Easy Examples

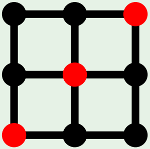
Theorem (Cameron, Kazanidis (2008))

If the quadrangle $Q(4,q)$[a] *possess an ovoid, then the corresponding classical group*[b] *is non-separating.*

[a] ...and some other quadrangles ...
[b] $O(5,q)$

Example (Order $(1,1)$ and $(2,1)$)





Lemma

A generalized quadrangle of order (s,t) *with* $s = 1$ *or* $t = 1$ *always possesses an ovoid.*

The Parabolic Quadric $Q(4, q)$

Corollary

An ovoid of $Q(4, q)$ *has size* $q^2 + 1$.

Some hyperplanes (**elliptic quadrics**) of $PG(4, q)$ contain exactly $q^2 + 1$ points of $Q(4, q)$!

Theorem (Ball, Govaerts, Storme (2005))

All ovoids of $Q(4, p)$, *p odd prime, correspond to elliptic quadrics.*

Idea.

Let $q = p^t$, p prime. A polynomial argument shows that each elliptic quadric intersects an ovoid of $Q(4, q)$ in 1 mod p points. If $t = 1$, then this gives a classification. □

The Elliptic Quadric $Q^-(5, q)$

Corollary

A (partial) ovoid of $Q^-(5, q)$ *has (at most) size* $q^3 + 1$.

Lemma (Thas (1981))

A partial ovoid $\mathcal{O}$ *of a generalized quadrangle of order* (q, q^2) *has at most size* $q^3 - q^2 + q$.

Proof.

- As the point graph is an SRG, the adjacency matrix of the point graph has three eigenspaces.
- One eigenspace V has dimension $q^3 - q^2 + q$.
- Let E be the orthonormal projection onto V.

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Proof.

- As the point graph is an SRG, the adjacency matrix of the point graph has three eigenspaces.
- One eigenspace V has dimension $q^3 - q^2 + q$.
- Let E be the orthonormal projection onto V.
- The submatrix $E_{\mathcal{O}}$ of E indexed by $\mathcal{O}$ has form $\alpha I + \beta J$.[a]
- The matrix $E_{\mathcal{O}}$ has full rank. □

[a] Here I is the identity matrix and J the all-ones matrix.

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Corollary

A (partial) ovoid of $Q^-(5, q)$ *has (at most) size* $q^3 + 1$.

Lemma (Thas (1981))

A partial ovoid $\mathcal{O}$ *of a generalized quadrangle of order* (q, q^2) *has at most size* $q^3 - q^2 + q$.

Theorem (De Beule, Klein, Metsch, Storme (2008))

A partial ovoid $\mathcal{O}$ *of* $Q^-(5, q)$ *has at most size* $(q^3 + q + 2)/2$.

This bound is tight for $q = 2, 3$.

Theorem (I., Sin, Xiang (2016))

A partial ovoid $\mathcal{O}$ *of* $Q^-(5, q)$, $q = p^t$, *p prime, has at most size*

$$q^{1+\log_p((2p^2+1)/3)}.$$

The Elliptic Quadric $Q^-(5, q)$

Theorem (De Beule, Klein, Metsch, Storme (2008))

A partial ovoid $\mathcal{O}$ of $Q^-(5, q)$ has at most size $(q^3 + q + 2)/2$.

Theorem (I., Sin, Xiang (2016))

A partial ovoid $\mathcal{O}$ of $Q^-(5, q)$, $q = p^t$, p prime, has at most size

$$q^{1+\log_p((2p^2+1)/3)}.$$

Proof.

Use p-ranks instead of ranks. □

For $p = 2$ fixed, we obtain the bound $\approx q^{2.59}$.
For $p = 3$ fixed, we obtain the bound $\approx q^{2.68}$.

Example

The largest known infinite family has size $\approx 3q^2/2$.

Intriguing Sets

Definition

A set of points $\mathcal{O}$ is an *m*-**ovoid** iff every line contains exactly *m* points.

Ovoids are 1-ovoids.

Definition

A set of points $\mathcal{T}$ is an *i*-**tight set** iff it is a linear combination of lines and $|\mathcal{T}| = i(s+1)$.

Lines are 1-tight sets.

Lemma

An i-tight set $\mathcal{T}$ and an m-ovoid $\mathcal{O}$ satisfy $|\mathcal{T} \cap \mathcal{O}| = mi$.

Tight sets and ovoids are **intriguing sets**.

Intriguing Sets – a Different Perspective

Recall: the point graph is strongly regular.
The adjacency matrix A of a SRG has three pw. orthogonal eigenspaces: $\langle \mathbf{j} \rangle$, V^+, V^-.

Lemma

- *A point set $\mathcal{O}$ is an m-ovoid iff its characteristic vector lies in $\langle \mathbf{j} \rangle + V^-$ and $|\mathcal{O}| = m(st+1)$.*
- *A point set $\mathcal{T}$ is an i-tight set iff its characteristic vector lies in $\langle \mathbf{j} \rangle + V^+$ and $|\mathcal{T}| = i(s+1)$.*

Now the following makes sense ...

Definition

A vector in $\langle \mathbf{j} \rangle + V^\epsilon$, $\epsilon \in \{-,+\}$, is a weighted intriguing set.

As V^+ and V^- are orthogonal, weighted m-ovoids and weighted i-tight sets have constant intersection mi.

Existence of m-Ovoids

Question

For which m possesses $Q^-(5, q)$ an m-ovoid?

Lemma (Repetitive . . .)

The elliptic quadric $Q^-(5, q)$ does not possess an ovoid.

Proof.

- Two non-adjacent points p_1, p_2 have $q^2 + 1$ common neighbors N.
- The vector $q\chi_{\{p_1,p_2\}} + \chi_N$ is a weighted $(q + 1)$-tight set $\mathcal{T}$.

Existence of m-Ovoids

Question

For which m possesses $\mathrm{Q}^-(5, q)$ an m-ovoid?

Lemma (Repetitive . . .)

The elliptic quadric $\mathrm{Q}^-(5, q)$ does not possess an ovoid.

Proof.

- Two non-adjacent points p_1, p_2 have $q^2 + 1$ common neighbors N.
- The vector $q\chi_{\{p_1,p_2\}} + \chi_N$ is a weighted $(q + 1)$-tight set $\mathcal{T}$.
- Suppose that $\mathcal{O}$ is a 1-ovoid.
- We can choose p_1, p_2 such that $p_1 \in \mathcal{O}$, $p_2 \notin \mathcal{O}$.
- $q + 1 = |\mathcal{O} \cap \mathcal{T}| = q|\{p_1, p_2\} \cap \mathcal{O}| + |N \cap \mathcal{O}| = q$. □

Existence of m-Ovoids

A more elaborate version can be used to obtain a proof of ...

Theorem (Segre (1965))

An m-ovoid of $\mathrm{Q}^-(5, q)$ has $m = (q+1)/2$.

The technique has other applications in various SRGs.

Theorem (Bamberg, Devillers, Schillewaert (2012))

If $\mathcal{O}$ is an m-ovoid of the generalized quadrangle $\mathrm{H}(4, q^2)$ of order (q^2, q^3), $q > 2$, then $m \gtrsim q^{5/2}$.

Problem (Famous, hard)

Does the generalized quadrangle $\mathrm{H}(4, q^2)^D$ possess an ovoid?

Computer Help

How else to use weighted intriguing sets?

Question

Let $\mathcal{O}$ be a point set. Is $\mathcal{O}$ an ovoid?

Problem

Testing all lines is too much work.

Computer Help

How else to use weighted intriguing sets?

Question

Let $\mathcal{O}$ be a point set. Is $\mathcal{O}$ an ovoid?

Problem

Testing all lines is too much work.

Answer

- *Let G be the automorphism group (hopefully very large).*
- *Take a largish subgroup S of G.*
- *A line ℓ is a 1-tight set, so $\sum_{s \in S} \chi_\ell^s$ is a weighted $|\ell^S|$-tight set.*

Sometimes this reduces the problem size! Computer can look for proofs!
Alternative proofs for the non-existence of ovoids in the geometries $\mathrm{H}(5,4)$ and $\mathrm{Q}^+(9,2^t)$: Bamberg, De Beule, I. (2016).

Computer Help – Example

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Nonexistence of ovoids in $Q^-(5,3)$:

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Nonexistence of ovoids in $\mathrm{Q}^-(5,3)$:

- Let S be the stabilizer of two non-adjacent points p_1, p_2.
- S has line orbits of lengths 20, 180 and 80.
- Four point orbits: $\{p_1, p_2\}$, O_1, O_2, the common neighborhood O_3.

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Answer

- *Let G be the automorphism group (hopefully very large).*
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Nonexistence of ovoids in $\mathrm{Q}^-(5,3)$:

- Let S be the stabilizer of two non-adjacent points p_1, p_2.
- S has line orbits of lengths 20, 180 and 80.
- Four point orbits: $\{p_1, p_2\}$, O_1, O_2, the common neighborhood O_3.
- We can assume that $p_1 \in \mathcal{O}$ and $p_2 \notin \mathcal{O}$, which forces $|\mathcal{O} \cap O_3| = 0$.
- The corresponding weighted 20-, 80- and 180-tight sets define the equations

$$20 = 10|\mathcal{O} \cap \{p_1, p_2\}| + |\mathcal{O} \cap O_1| + |\mathcal{O} \cap O_3| = 10 + |\mathcal{O} \cap O_1|$$
$$180 = 9|\mathcal{O} \cap O_1| + 6|\mathcal{O} \cap O_2|$$
$$80 = 4|\mathcal{O} \cap O_2| + 8|\mathcal{O} \cap O_3| = 4|\mathcal{O} \cap O_2|$$

No integer solution! Ovoid does not exist!

The Symplectic Space W(3, q)

The quadrangle W(3, q) has order (q, q) and is the dual of Q(4, q).

- Common neighborhood of p_1, p_2 non-adjacent: $\{p_1, p_2\}^{\sim}$. Size: $q + 1$.
- The set $\{p_1, p_2\}^{\sim\sim} = \{p_1, p_2, \ldots\}$ has size $q + 1$.
- The set $\chi_{\{p_1,p_2\}^{\sim}} + \chi_{\{p_1,p_2\}^{\sim\sim}}$ is a 2-tight set.
- For ovoid $\mathcal{O}$: either $|\mathcal{O} \cap \{p_1, p_2\}^{\sim}| > 0$ or $|\mathcal{O} \cap \{p_1, p_2\}^{\sim\sim}| > 0$.

The Symplectic Space $W(3, q)$

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- The set $\{p_1, p_2\}^{\sim\sim} = \{p_1, p_2, \ldots\}$ has size $q + 1$.
- The set $\chi_{\{p_1,p_2\}^{\sim}} + \chi_{\{p_1,p_2\}^{\sim\sim}}$ is a 2-tight set.
- For ovoid $\mathcal{O}$: either $|\mathcal{O} \cap \{p_1, p_2\}^{\sim}| > 0$ or $|\mathcal{O} \cap \{p_1, p_2\}^{\sim\sim}| > 0$.

Theorem (Thas)

The generalized quadrangle $W(3, q)$ possess an ovoid only if q even.

Actually, it is if and only if:

Lemma

If q even, then $W(3, q) \cong Q(4, q)$.

In particular, $W(3, 2^t)$ contains these ovoids:

- The elliptic quadric $Q^-(3, 2^t)$.
- The Suzuki-Tits ovoid if t odd.

Existence of Tight Sets

In some settings the existence of *i*-tight sets is interesting.

- Consider the point graph of $Q^+(5, q)$, a generalization of the GQ $Q^+(3, q)$.
- *i*-tight sets of $Q^+(5, q)$ are known as Cameron-Liebler line classes.

Conjecture (Cameron, Liebler (1982))

An i-tight set of $Q^+(5, q)$ *exists if and only if* $i \in \{1, 2\}$.

Existence of Tight Sets

In some settings the existence of i-tight sets is interesting.

- Consider the point graph of $\mathrm{Q}^+(5, q)$, a generalization of the GQ $\mathrm{Q}^+(3, q)$.
- i-tight sets of $\mathrm{Q}^+(5, q)$ are known as Cameron-Liebler line classes.

Conjecture (Cameron, Liebler (1982))

An i-tight set of $\mathrm{Q}^+(5, q)$ exists if and only if $i \in \{1, 2\}$.

Today several infinite families with $i \approx q^2/2$ are known.

Theorem (Metsch (2014))

An i-tight set of $\mathrm{Q}^+(5, q)$ with $i > 2$ exists only if $i \geq cq^{4/3}$.

More complicated modular condition by Gavrilyuk, Metsch (2014).
Both proofs use a particular m-ovoid.

Generalized Hexagons

Definition

Let $\mathcal{P}$ (points) and $\mathcal{L}$ (lines) be disjoint sets. Let $\mathbf{I} \subseteq \mathcal{P} \times \mathcal{L}$ an **incidence relation**. The structure $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a generalized hexagon of order (s, t) if and only if

(I) Each line contains exactly $s + 1$ points.

(II) Each point lies in exactly $t + 1$ lines.

(III) The **incidence graph** has girth 12 and diameter 6. (Moore Graph!)

Example (Order $(1, 2)$)

The incidence graph of the Fano plane:

- Fano Points: 001, 010, 011, 100, 101, 110, 111.
- Fano Lines: $\{001, 010, 011\}$, $\{001, 100, 101\}$, $\{001, 110, 111\}$, $\{010, 100, 110\}$, ...

These are our 14 points. The lines of the hexagon are the $7 \times 3 = 21$ flags of the Fano plane, e.g. $(001, \{001, 010, 011\})$, $(010, \{001, 010, 011\})$ ...

Existence

Lemma (Duality)

If $(\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a generalized hexagon of order (s, t), then $(\mathcal{L}, \mathcal{P}, \mathbf{I})$ is a generalized hexagon of order (t, s).

There are only four known families of generalized hexagons with $1 < s \leq t$:

- The split Cayley hexagon $\mathrm{H}(q)$ of order (q, q) and its dual $\mathrm{H}(q)^D$.
- The twisted triality hexagon $\mathbf{T}(q^3, q)$ of order (q^3, q) and its dual $\mathbf{T}(q, q^3)$.

Iff q is a power of 3, then $\mathrm{H}(q)$ and $\mathrm{H}(q)^D$ are isomorphic.

Problem (Very, very, very, very hard.)

Are there any other generalized hexagons?[a]

[a] Same question for projective planes = generalized triangles, generalized quadrangles and generalized octagons. Find a new one and be famous!

Ovoids of Generalized Hexagons

Definition

A 1-**ovoid** of a GH is a point set, which meets every lines exactly once.

Theorem (De Bruyn, Vanhove (2013))

The dual twisted triality hexagon has no 1*-ovoid.*

- There exist 1-ovoids for $\mathrm{H}(q)$ and $q = 2, 3, 4$.
- There exists no 1-ovoid for $\mathrm{H}(2)^D$.
- There exists a 1-ovoid for $\mathrm{H}(3)^D \cong \mathrm{H}(3)$.

Next open case: $\mathrm{H}(4)^D$.

1-Ovoids of $\mathrm{H}(q)^D$

Definition

A 1-**ovoid** of a GH is a point set, which meets every lines exactly once.

- The hexagon $\mathrm{H}(q)^D$ has $|\mathcal{P}| = (q^2+q+1)(q^3+1) = |\mathcal{L}|$.
- A 1-ovoid has size q^4+q^2+1.

The point graph of $\mathrm{H}(q)^D$ is distance-3-regular, so its adjacency matrix has four eigenspaces $\langle \mathbf{j} \rangle$, V_1, V_2, V_3.

Lemma

The characteristic vector of a 1-ovoid of $\mathrm{H}(q)^D$ *is in* $\langle \mathbf{j} \rangle \perp V_3$.

We can look for good vectors in $\langle \mathbf{j} \rangle \perp V_1 \perp V_2$!

A Good Tight Set

The dual split Cayley hexagon $\mathsf{H}(q)^D$ contains a subhexagon $\mathsf{H}(q,1)$.

Lemma

The point set of a $\mathsf{H}(q,1) \subseteq \mathsf{H}(q)^D$ *is a* (q^2+q+1)*-tight set.*

A 1-ovoid of $\mathsf{H}(q,1)$ has size q^2+q+1, so ...

Corollary

If $\mathcal{O}$ *is a* 1*-ovoid of* $\mathsf{H}(q)^D$*, then* $\mathcal{O} \cap \mathsf{H}(q,1)$ *is a* 1*-ovoid of* $\mathsf{H}(q,1)$*.*

One can use this to show the following:

Theorem (Bishnoi, I. (unpublished))

The dual split Cayley heaxgon $\mathsf{H}(4)^D$ *does not possess a* 1*-ovoid.*

Computer proof!

Basic idea:

(1) Classify all 1-ovoids of $H(4,1)$.

(2) Show for all 1-ovoids of $H(4,1)$ that they do not extend to an ovoid of $H(4)^D$.

Computer proof!

Basic idea:

(1) Classify all 1-ovoids of $H(4,1)$.

(2) Show for all 1-ovoids of $H(4,1)$ that they do not extend to an ovoid of $H(4)^D$.

In detail:

(1) 1-ovoids of $H(4,1)$ are perfect matchings in the incidence graph of $PG(2,4)$. There are $18534400 \approx 2 \cdot 10^7$ of them.

(2) Knuth's Dancing Links algorithm can iterate through 1-ovoids of $H(4,1)$.

(3) Use GAP's group functionality, so that we skip most of the $2 \cdot 10^7$ iterations.

(4) One obtains 350 non-isomorphic 1-ovoids of $H(4,1)$.

(5) Use ILP solvers to show that they do not extend to an ovoid of $H(4)^D$. □

1-ovoids of $H(4)^D$ do not exist!

Thank you for your attention!