The Manickam-Miklós-Singhi Conjecture for Vector Spaces

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Irsee 2014

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History

- 1984: Bier defines the *i*-**th distribution invariant** of an **association scheme**.
- 1987: Bier and Delsarte generalize this concept to **distribution numbers** of association schemes.
- 1986: Manickam, a student of Eiichi Bannai, publishes his PhD thesis on "Distribution Invariants of Association Schemes".
- 1988: Manickam, Miklós, and Singhi publish the Manickam-Miklós-Singhi conjecture for sets and vector spaces.
- 2014: Simeon Ball tells me about this conjecture.[1]

[1] This important fact was unfortunately missing in the actual talk given in Irsee by the speaker.

The MMS Conjecture for Sets

1. Consider $M = \{1, \ldots, 10\}$.
2. Let $f : M \to \mathbb{R}$ a **weighting** of M with $\sum_{x \in M} f(x) = 0$.

Question

How many subsets S of M have nonnegative weight, i.e. how many such S satisfy $\sum_{x \in S} f(x) \geq 0$?

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Answer

At least 2^9. If $S \subseteq M$ has negative weight, then its complement $\complement S$ has positive weight.

Too simple. Let's change the question...

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How many 3-element subsets S of M have nonnegative weight, i.e. how many such S satisfy $\sum_{x \in S} f(x) \geq 0$?

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Answer (Marino, Chiaselotti (2002), Hartke, Stolee (2014))

At least $\binom{7}{3} = 35$.

Some Examples

We have $\binom{10}{3} = 120$ subsets with 3 elements.

Example

Put the weight 1 on $1, \ldots, 9$ and the weight -9 on 10. Then we have $\binom{9}{3} = 84$ nonnegative 3-element subsets.

Example

Put the weight -1 on $1, \ldots, 9$ and the weight 9 on 10. Then we have $\binom{9}{2} = 36$ nonnegative 3-element subsets.

Example

Put the weight 3 on $1, \ldots, 7$ and the weight -7 on $8, 9, 10$. Then we have $\binom{7}{3} = 35$ nonnegative 3-element subsets.

The last example is the unique smallest example (Marino, Chiaselotti, Hartke, Stolee).

The Manickam-Miklós-Singhi Conjecture for Sets

Conjecture (Manickam-Miklós-Singhi)

Let $M = \{1, \dots, n\}$, $n \geq 4k$, and $f : M \to \mathbb{R}$ a weighting with $\sum_{x \in M} f(x) = 0$. Then the set Y of nonnegative k-element subsets of M satisfies

$$|Y| \geq \binom{n-1}{k-1}.$$

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Authors	**Year**	**Bound on** n	$k = 10$
Bier, Manickam	1987	$\approx k^{2k+1}$	$4 \cdot 10^{19}$
Manickam, Miklós	1988	$(k-1)(k^k + k^2) + k$	$9 \cdot 10^{10}$
Bhattacharya	2003	$2^{k+1} e^k k^{k+1}$	$5 \cdot 10^{18}$
Tyomkyn	2012	$k^2 (4e \log k)^k$	10^{16}
Alon, Huang, Sudakov		$\min\{33k^2, 2k^3\}$	2000
Frankl	2013	$\frac{3}{2} k^3$	1500
Chowdhury, Sarkis, Shahriari	2014	$8k^2$	800
Pokrovskiy	201?	$10^{46} k$	10^{47}

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One can also try to solve the problem for small k.

Authors	**Year**	k
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Blinovsky, 201?: complete solution?

Vector Spaces

- A vector space of dimension n over a finite field with q elements: V.
- S is a subspace of V of dimension k: k-space.
- 1-spaces: **points**, $(n-1)$-spaces: **hyperplanes**, $\mathcal{P}$: all points of V.
- The number of k-spaces in an n-space: $\begin{bmatrix} n \\ k \end{bmatrix}$.

Conjecture (Manickam-Miklós-Singhi)

Let $n \geq 4k$, and $f : \mathcal{P} \to \mathbb{R}$ a weighting with $\sum_{x \in \mathbb{P}} f(x) = 0$. Then the set Y of nonnegative k-spaces of V satisfies

$$|Y| \geq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Some Examples

Example

Let P be a point. Put the weight $\begin{bmatrix} n \\ 1 \end{bmatrix} - 1$ on P, and -1 on all the other points. Then exactly the $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ k-spaces through P have nonnegative weight.

Example

Let H be a hyperplane. Put the weight -1 on all points not in H, and $q^{n-1}/\begin{bmatrix} n-1 \\ 1 \end{bmatrix}$ on all points in H. Then exactly the $\begin{bmatrix} n-1 \\ k \end{bmatrix}$ k-spaces in H have nonnegative weight.

A Strengthend Conjecture

Conjecture

Let $n \geq k$, and $f : \mathcal{P} \to \mathbb{R}$ a weighting with $\sum_{x \in \mathbb{P}} f(x) = 0$. Then the set Y of nonnegative k-spaces of V satisfies

$$|Y| \geq \min\left\{ \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \begin{bmatrix} n-1 \\ k \end{bmatrix} \right\}$$

with equality if and only if Y is either the set of all k-spaces through a point or the set of all k-spaces in a hyperplane.

It is enough to show the conjecture for $n \geq 2k$ as the case $n < 2k$ follows from duality.

More History

The conjecture is true if k divides n.

Theorem (Manickam and Singhi (1988))

If k divides n, then the smallest set of nonnegative k-spaces Y is an **Erdős-Ko-Rado set** *of maximum size, i.e. a set of pairwise non-trivially intersecting k-spaces of maximum size.*

More History

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Proof.

As k divides n, there exists a spread $\mathcal{S}$ of $\mathcal{P}$ into k-spaces. Let S be a k-space with negative weight. Suppose $S \in \mathcal{S}$. We have

$$\sum_{x \in M} f(x) = \sum_{T \in \mathcal{S}} \sum_{x \in T} f(x) = 0,$$

so at least one element $T \in \mathcal{S} \setminus \{S\}$ has positive weight. Double counting over all $\mathcal{S}$ with $S \in \mathcal{S}$ shows $|Y| \geq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ with equality if and only if Y is an Erdős-Ko-Rado set of maximum size. □

Contemporary History

Theorem (Chowdhury, Sarkis, Shahriari (2014))

If $n \geq 3k$, then

$$|Y| \geq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

with equality if and only if Y is the set of all k-spaces through a fixed point.

Theorem (Huang, Sudakov (2014))

If $n \geq ck$ for sufficiently large c, then

$$|Y| \geq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Two Ideas

The ideas used by Chowdhury, Sarkis, and Shahriari. Many of the following only holds for $n \geq 2k+1$.

1. If k does not divide n, then one can still use something similar to a spread to imitate the Manickam-Singhi double count. This shows
$$|Y| \geq (1 - O(1/q)) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$
2. An eigenvalue trick shows
$$|Y| \geq (1 - O(1/q)) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Combining both ideas shows the result for $n \geq 3k$ and $q \geq 2$.

The First Idea

Theorem (Beutelspacher (1975))

Let $n = rk + \delta$, $r \in \mathbb{Z}$, $\delta < k$. Then one can partition $\mathcal{P}$ into one $(k+\delta)$-space and k-spaces.

Chowdhury, Sarkis, and Shahriari use this to show that if there exists a k-space S with negative weight, then there are

$$\left(1 - \frac{2}{q^{n-2k-\delta+1}}\right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

k-spaces with positive weight which intersect S trivially.

The Second Idea

- Let W be the **incidence matrix** whose rows are indexed by the k-spaces and whose columns are indexed by the points, i.e.

$$W_{PS} = \begin{cases} 1 & \text{if } P \text{ is a point of } S, \\ 0 & \text{otherwise.} \end{cases}$$

- Let A be the **distance-**$(k-1)$**-adjacency matrix** of k-spaces, i.e. the symmetric matrix indexed by k-spaces with

$$A_{ST} = \begin{cases} 1 & \text{if } \dim(S \cap T) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

- If we view the weight function f as a vector, then it is well-known that $b = Wf$ is an **eigenvector** of A, i.e. the vector b indexed by the k-spaces with the weights of the k-spaces as its entries is an eigenvector of A.

The Second Idea

- We know that the weight vector b of the k-spaces is an eigenvector of the distance-$(k-1)$-adjacency matrix A with **eigenvalue** λ. This shows for a k-space C

$$\sum_{\dim(S\cap C)=1} b_S = (Ab)_C = \lambda b_C.$$

- If C is a **highest weight** k-space, then this shows that at least λ k-spaces, which meet C in exactly a point, have nonnegative weight. Fortunately,

$$\lambda \geq \left(1 - \frac{3}{q^{n-2k+1}}\right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Both Ideas Together

Recall $n = rk + \delta$.

- For each negative k-space, there are

$$\left(1 - \frac{2}{q^{n-2k-\delta+1}}\right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

  nonnegative k-spaces disjoint to this k-space.
- The highest weight k-space meets at least

$$\left(1 - \frac{3}{q^{n-2k+1}}\right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

  nonnegative k-spaces in a point.
- This shows the conjecture for $n \geq 3k$ and $q \geq 2$.
- Similar arguments: $n \geq 2k$ and q large (I., submitted).
- Similar arguments: $(n, k) = (5, 2)$ and $q \geq 2$ (Chowdhury, Shahriari, Sarkis, unpublished(?)).

Generalizations

Other incidence geometries. For **polar spaces** of rank d, the Manickam-Singhi technique shows the following.

Theorem

If there exists a spread $\mathcal{S}$ of (totally isotropic/singular) k-spaces, then the set of nonnegative k-spaces Y has at least size $n/|\mathcal{S}|$. Here n is the number of k-spaces. In case of equality Y is an Erdős-Ko-Rado set.

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Non-trivial results seem to be hard ...

1. Spreads are only known for $k = d$ and some other special cases. What are good substitutes?
2. The presented eigenvalue trick does not always work. Is there another one?
3. Combine both results?

Thank You!