Large Intersecting Families in Finite Vector Spaces

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The Erdős-Ko-Rado Theorem (Sets)

Sets:

Theorem (Erdős, Ko, Rado (1961))

For $n \geq 2k$, an intersecting family $\mathcal{F}$ of k-sets in $\{1, \ldots, n\}$ satisfies

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

For $n \geq 2k+1$ equality holds if and only if ***$\mathcal{F}$ consists of all k-sets that contain a fixed element.***

There is no structure for $n = 2k$.

The Erdős-Ko-Rado Theorem (Vector Spaces)

Vector Spaces:

Theorem (Hsieh (1975); Frankl, Wilson (1986); Godsil, Newman (2006))

For $n \geq 2k$, an intersecting family of k-spaces in $\mathbb{F}_q^n$ satisfies

$$|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Equality holds if and only if either (1) $\mathcal{F}$ consists of all k-spaces that contain a fixed 1-space, or ***(2) $n = 2k$ and $\mathcal{F}$ consists of all k-space in a fixed hyperplane.***

$$\begin{bmatrix} a \\ b \end{bmatrix} = \prod_{i=1}^{b} \frac{q^{a-i+1}-1}{q^i-1}$$

is the number of b-spaces in $\mathbb{F}_q^a$.

The Hilton-Milner Theorem (Sets)

$\gamma(\mathcal{F})$ = **diversity** of $\mathcal{F}$ = # members not in the most popular element.

Sets:

Theorem (Hilton, Milner (1967))

For $n \geq 2k+1$ and $\gamma(\mathcal{F}) \geq 1$, an intersecting family $\mathcal{F}$ of k-sets in $\{1, \ldots, n\}$ satisfies

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Equality holds if $\mathcal{F}$ is one of the following:

- $\mathcal{F} = \{F \text{ } k\text{-set} : x \in F, |F \cap Y| \geq 1\} \cup \{Y\}$ *for a fixed element x and a fixed k-set Y with $x \notin Y$.*
- ***$k = 3$ and $\mathcal{F}$ consists of all k-sets meeting a fixed 3-set in a ≥ 2-set.***

The Hilton-Milner Theorem (Vector Spaces)

Vector Spaces:

Theorem (Blokhuis et al. (2010), Wang et al. (2023))

For $n \geq 2k+1$ and $\gamma(\mathcal{F}) \geq 1$, an intersecting family $\mathcal{F}$ of k-spaces in $\mathbb{F}_q^n$ satisfies

$$|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + q^k.$$

Equality holds if $\mathcal{F}$ is one of the following:

- $\mathcal{F} = \{F \text{ } k\text{-space} : X \subset F, \dim(F \cap Y) \geq 1\} \cup \{Y\}$ *for a fixed 1-space X and a fixed k-space Y with $\dim(X \cap Y) = 0$.*
- $k = 3$ *and $\mathcal{F}$ consists of all k-spaces meeting a fixed 3-space in a ≥ 2-space.*

But what about $n = 2k$?

Hilton-Milner Theorem (Vector Spaces), $n = 2k$

For $n = 2k$, the largest ones have size $\begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix} > q^{k(k-1)}$.

Theorem (Blokhuis, Brouwer, Szőnyi (2010))

For $k < q \log q - q$ and $\gamma(\mathcal{F}) \geq 1$,[a] an intersecting family $\mathcal{F}$ of k-spaces in $\mathbb{F}_q^{2k}$ satisfies $|\mathcal{F}| \leq \frac{1}{2} q^{k(k-1)}$.

[a] Here $\gamma(\mathcal{F})$ denotes the number of members of $\mathcal{F}$ not incident with the **most popular point/hyperplane**.

Likely truth: $\approx \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} 2k-2 \\ k-1 \end{bmatrix}$.

Theorem (Ih. (2019))

For $k \geq 5$, $q \geq 3$, and $\gamma(\mathcal{F}) \geq 1$, an intersecting family $\mathcal{F}$ of k-spaces in $\mathbb{F}_q^{2k}$ satisfies $|\mathcal{F}| \leq (1 + 3q^{-1}) \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} 2k-2 \\ k-1 \end{bmatrix}$.

Hilton-Milner for vector spaces **not yet complete**!

The Complete Intersection Theorem (Sets)

Frankl families:

$$\mathcal{A}(n,k,t,s) = \{A : |A| = k, |A \cap \{1,\dots,t+2s\}| \geq t+s\}.$$

Put $AK(n,k,t) = \max_{s \geq 0} |\mathcal{A}(n,k,t,s)|$.

Frankl conjectured that these are the largest.

Theorem (Ahlswede, Khatchatrian (1997))

For a t-intersecting family of k-sets, $|\mathcal{F}| \leq AK(n,k,t)$.

Equality holds, up to a few exceptions, only if $\mathcal{F}$ is ***isomorphic to a Frankl family****.*

The Complete Intersection Theorem (Vector Spaces)

Vector Spaces:

Theorem (Frankl, Wilson (1986); Tanaka (2006))

For $n \geq 2k$, a t-intersecting family of k-sets in $\mathbb{F}_q^n$ satisfies

$$|\mathcal{F}| \leq \max\left(\begin{bmatrix} n-t \\ k-t \end{bmatrix}, \begin{bmatrix} 2k-t \\ k \end{bmatrix}\right).$$

Equality holds if $\mathcal{F}$ is one of the following:

- *all k-spaces through a fixed t-space,*
- *$n = 2k$ and all k-spaces in a fixed $(2k-t)$-space.*

Note that $n \geq 2k$ is not a restriction by duality.

Frankl Degree-Type Theorem (Sets)

Recall that $\gamma(\mathcal{F})$ denote the number elements of $\mathcal{F}$ not containing a **most popular element**.

Theorem (Frankl (1987), Kupavskii, Zakharov (2018))

If $\gamma(\mathcal{F}) \geq \binom{n-u-1}{k-u}$ for some real $3 \leq u \leq k$, then

$$|\mathcal{F}| \leq \binom{n-1}{k-1} + \binom{n-u-1}{k-u} - \binom{n-u-1}{k-1}.$$

Equality holds only if u is an integer and

$$\mathcal{F} = \{F : x \in F, |F \cap Y| \geq 1\} \cup \{F : Y \subset F\}$$

for some element x and some u-set Y.

Frankl Degree-Type Theorem (Vector Spaces)

Theorem (Ih., Kupavskii (soon))

Let $n \geq 2k+1$. Assume that for some $i = 4, \dots, k$ we have

$$\gamma(\mathcal{F}) \geq q^k \begin{bmatrix} n-i-1 \\ k-i \end{bmatrix}.$$

Under mild conditions in n and k, or q, equality occurs only if

$$\mathcal{F} = \{F : X \subset F, \dim(F \cap Y) \geq 1\} \cup \{F : \boldsymbol{Y \subset X+F}\}.$$

Problems:

- Not for $i = 3$ (the "2 out of 3"-family has the same size).
- Only for integers.
- **Only for n $\geq$2k+1.**

Largest Diversity

Theorem (Frankl, Wang (2024))

*For $n > 36k$, the "2 out of 3" family has the **largest diversity** among all intersecting families of k-sets in $\{1, \ldots, n\}$.*

Theorem (Ih., Kupavskii (soon))

*For $n \geq 2k + 1$ and some mild conditions, the "2 out of 3" family has the **largest diversity** among all intersecting families of k-spaces in $\mathbb{F}_q^n$.*

Largest Diversity

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What about $n = 2k$? Again, γ needs to be defined differently.

Maybe:

- fix 3-space X and take all k-spaces containing 2-space of X; or
- fix $(2k-3)$-space X, take all k-spaces in $(2k-2)$-space through X.

The Structure of Large Intersecting Families (Sets)

Frankl, Füredi in the 1970s and 1980s: **structural results**.

One Example:

Theorem (Frankl (1978))

For $\varepsilon > 0$ and positive integers n, k, t, s such that $t > 2s(s-1)$ and $\varepsilon < \varepsilon(t,s) < 1/t(t+2s)$, let $\mathcal{F}$ be a t-intersecting family of k-sets in $\{1,\ldots,n\}$. ***Suppose that each element of $\{1,\ldots,n\}$ lies in less than $([(t+s)/(t+2s)]+\varepsilon)|\mathcal{F}|$ elements of $\mathcal{F}$****, $\mathcal{F}$ is maximal under these constraints, and $n > n_0(k,s,t)$. Then*

$$\mathcal{F} \cong \mathcal{A}(n,k,t,s).$$

Dinur and Friedgut (2009): **junta method**.

Kupavskii and Zakharov (2018) and subsequent work: **peeling method**.

What about **vector spaces**?

The Structural Result, Examples on Blackboard

Asymptotically **tight** in any **reasonable asymptotic regime**:

Theorem (Ih., Kupavskii (soon))

*Fix $n, k, t \geq 0$ and $s \geq 1$ such that $k \geq s+t+1$ and **n≥2k+s+1**. Let $\mathcal{F}$ be a t-intersecting family of k-spaces in $\mathbb{F}_q^n$. Then there exists a **t-intersecting family $\mathcal{C}_{s+t}$ of subspaces of dimension at most $s+t$** such that the number subspaces in $\mathcal{F}$ do not contain a subspace in $\mathcal{C}_{s+t}$ is bounded. Specifically,*

$$|\mathcal{F} \setminus \mathcal{F}[\mathcal{C}_{s+t}]| \leq C_q q^{(k-t)(n-k)-(s+1)(n-k-t-s-1)}.$$

where $C_q \to 1$ for $q \to \infty$.

Here

$$\mathcal{C}[\mathcal{X}] := \{S : S \in \mathcal{C}, X \subset S \text{ for some } X \in \mathcal{X}\}.$$

Spread Approximation Method

The **spread approximation method** (Kupavskii, Zakharov (2024)) has two parts:

- the **sunflower lemma** (Alweiss, Lovett, Wu Zhang (2021)),
- the **peeling procedure**.

[1] Unlike for sets, the trivial upper bound on sunflower-free sets is almost tight.

Spread Approximation Method

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- the **sunflower lemma** (Alweiss, Lovett, Wu Zhang (2021)),
- the **peeling procedure**.

The first part does not work for vector spaces.

Ih., Kupavskii (2026): no resonable **sunflower lemma**.[1]

Two definitions are feasible, so there are two Erdős-Rado sunflower problems in vector spaces. See Etzion and Raviv (2015).

But **peeling** works with a definition of **subspace-spreadness**.

[1] Unlike for sets, the trivial upper bound on sunflower-free sets is almost tight.

Future Work

One can always list many EKR problems.

But, regarding vector spaces, I usually have one gripe:

Problem

Do everything for $n = 2k$.

This is the by far **hardest** and, therefore, **most interesting case**.

- For EKR for sets, $n \geq Ck$ and $n \geq k^2$ are very different.
- For EKR for vector spaces, $n \geq 2k+1$ is often like $n \geq k^2$.
- Thus, the ambition should always be $n = 2k$ or $n \geq 2k+1$.

One More: The Erdős-Matching Conjecture

Conjecture (Erdős (1965))

Let $\mathcal{F}$ be a largest family of k-sets in $\{1,\ldots,n\}$, no $s+1$ pairwise disjoint. Then

$$|\mathcal{F}| \leq \max\left(\binom{k(s+1)-1}{k}, \binom{n}{k} - \binom{n-s}{k}\right).$$

Frankl, Kupavskii and others have nice results!

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Conjecture (Ihringer (2021))

Let $\mathcal{F}$ be a largest family of k-spaces in $V(n,q)$ such that no $s+1$ are pairwise disjoint. Then $\mathcal{F}$ or $\overline{\mathcal{F}}$ is the union of s intersecting families.

Heering (2026):
Application to **EKR for flags** of type $\{k, k+1\}$ in $V(2k+1, q)$.

Shangguan Chong (上官冲) will present an **alternative generalization!**

Thank you for your attention!