A Complete Solution to the Erdös Matching Problem for Permutations

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Abstract

Recently, Inozemtsev, Kolupaev, and Kupavskii investigated families of permutations acting on a set of size n without matchings of size s. We give a complete classification of the largest such families (and for perfect matchings of K_{2n}).

1 Introduction

Remark Below is a note that I wrote on 4th August 2026 when discussing variants of the Erdős-Matching Conjecture with Nathan Lindzey and Jian Wang. Due to the publication of [1], I decided to put this note online.

For a family $\mathcal{F}$ of objects for which some notion of intersection is defined, let $\nu(\mathcal{F})$ denote its matching number, that is, the maximum number of pairwise disjoint elements of $\mathcal{F}$. We denote the permutations on $\{1, \ldots, n\}$ by $\mathrm{Sym}(n)$ Erdős asked about the largest family $\mathcal{F}$ of k-sets in $\{1, \ldots, n\}$ such that $\nu(\mathcal{F}) < s$ [5]. In particular, the famous *Erdős Matching Conjecture* asks if the largest family is always either the family of all k-sets that meet a fixed set of size $s-1$, or all k-sets in a fixed $ks-1$ set. Recently, Inozemtsev, Kolupaev, and Kupavskii investigated the analogous problem for permutations [7].

Theorem 1.1 (Inozemtsev, Kolupaev, and Kupavskii [7])**.** *Let n, s be integers with $s \leq \frac{n}{2^{17} \log n}$. If $\mathcal{F} \subseteq \mathrm{Sym}(n)$ and $\nu(\mathcal{F}) < s$, then $|\mathcal{F}| \leq (s-1)(n-1)!$. Equality holds if and only if $\mathcal{F}$ is the disjoint union of $s-1$ 1-cosets.*

The aim of this note is to characterize equality for all s. The authors in [7] also provide a stability result, something unfeasible with the technique presented in the present note.

Theorem 1.2. *If $\mathcal{F} \subseteq \mathrm{Sym}(n)$ and $\nu(\mathcal{F}) < s$, then $|\mathcal{F}| \leq (s-1)(n-1)!$. Equality holds if and only if $\mathcal{F}$ is the disjoint union of $s-1$ 1-cosets.*

More specifically, $\pi \in \mathcal{F}$ either depends on $\pi(i) \in S$ for some i and some S with $|S| = s-1$, or it depends on some $\pi^{-1}(j) \in T$ for some j and some T with $|T| = s-1$.

The same argument works for perfect matchings.

Theorem 1.3. *If $\mathcal{F}$ is a family of perfect matchings of K_{2n} with $\nu(\mathcal{F}) < s$, then $|\mathcal{F}| \leq (s-1) \cdot (2(n-1)-1)!!$. Equality holds if and only if $\mathcal{F}$ is determined by $s-1$ pairwise intersecting edges.*

More specifically, either there exists a number i and a set of $s-1$ numbers $j_1, \ldots, j_{s-1}$ such that $\mathcal{F}$ is the union of all perfect matchings that contain one of $\{i, j_\ell\}$, or $s = 4$ and $\mathcal{F}$ is the union of all perfect matchings that contain one element of some triangle $\{i, j\}$, $\{j, k\}$, $\{k, i\}$.

2 Preliminaries

Let X be a finite set. An *association scheme* with d classes is a pair $(X, \mathcal{R})$, where $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ is a set of d symmetric relations on X such that

1. $\mathcal{R}$ is a partition of $X \times X$;
2. R_0 is the identity relation;
3. There are constants p_{ij}^k such that for any pair $(x, y) \in R_k$ the number of $z \in X$ with $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k.

Put $v = |X|$. Each relation R_i defines a graph. Define symmetric $0, 1$-matrices A_i in $\mathbb{R}^{v \times v}$ by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

We denote the identity matrix by I, the all-ones matrix by J, and the all-ones vector by j. Clearly, $A_0 = I$ and $\sum A_i = J$. Also, since $A_i A_j = \sum_{k=0}^d p_{ij}^k A_k$ and $A_i = A_i^T$, the A_i commute. Thus, we can diagonalize them simultaneously and obtain a decomposition of $\mathbb{R}^v$ into $d+1$ pairwise orthogonal eigenspaces V_j of dimension f_j for $0 \leq j \leq d$. By convention, $V_0 = \langle j \rangle$ and $f_0 = 1$. Let E_j be the orthogonal projection onto V_j. Note that $E_0 = \frac{1}{v} J$. Call two vectors $\chi, \psi \in \mathbb{R}^v$ are *design-orthogonal* if $\chi^T E_j \chi \cdot \psi^T E_j \psi = 0$ for all $j \in \{1, \ldots, d\}$.

We will work with the permutation association scheme and the perfect matching association scheme, see [6], respectively, [8] for a more detailed discussion. Note that for both schemes there exists a group G that acts transitively on all pairs of pairs $(x, y), (x', y') \in R_i$ (for all i), that is, $(x, y)^g = (x', y')$ for some $g \in G$. In other words, G acts transitively on arcs of R_i. The following is well-known, for instance, see [3, Theorem 6.8].

Theorem 2.1. *Let G acting on $X \times X$ with orbits $\mathcal{R}$. Let $\mathcal{F}$ and $\mathcal{G}$ be subsets of X with characteristic vectors χ, respectively, ψ. Then χ and ψ are design-orthogonal if and only if $|\mathcal{F} \cap \mathcal{G}^g|$ is independent of $g \in G$.*

3 Proofs

Here we prove our main results.

Proof of Theorem 1.2. Let $\mathcal{G}$ be a set of n pairwise nonintersecting permutations, e.g., a cyclic subgroup of $\mathrm{Sym}(n)$ generated by an n-cycle. By averaging, it follows that $|\mathcal{F}| \leq (s-1)(n-1)!$ with equality if and only if all such $\mathcal{G}$ intersect $\mathcal{F}$ in exactly $s-1$ elements. It is shown in Lemma 5.1 in [6] that for each $j \in \{2, \dots, d\}$ (for an appropriate labelling of the V_j) there exists a $\mathcal{G}$ such that its characteristic vector ψ satisfies $\psi^T E_j \psi > 0$. Thus, by Theorem 2.1, the characteristic vector of χ of $\mathcal{F}$ lies in $\langle j \rangle + V_1$. All $0, 1$-vectors in $\langle j \rangle + V_1$ have been classified in Corollary 2 in [4]. $\square$

The proof for perfect matchings is almost the same.

Proof of Theorem 1.3. Let $\mathcal{G}$ be a set of $2n-1$ pairwise nonintersecting perfect matchings of K_{2n}. By averaging, it follows that $|\mathcal{F}| \leq (s-1)(2(n-1)-1)!!$ with equality if and only if all such $\mathcal{G}$ intersect $\mathcal{F}$ in exactly $s-1$ elements. It is shown in Theorem 5.1 in [8] that for each $j \in \{2, \dots, d\}$ (for an appropriate labelling of the V_j) there exists a $\mathcal{G}$ such that its characteristic vector ψ satisfies $\psi^T E_j \psi > 0$. Thus, by Theorem 2.1, the characteristic vector of χ of $\mathcal{F}$ lies in $\langle j \rangle + V_1$. All $0, 1$-vectors in $\langle j \rangle + V_1$ have been classified in Theorem 6.2 in [2]. The case that $\mathcal{F}$ is the complement of a triangle can be easily ruled out. $\square$

References

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